Location Routing Problem with Time Dependent Travel Times

Shokoufeh Mirzaei
Wichita State University, Wichita, KS, USA
Krishna Krishnan*
Wichita State University, Wichita, KS, USA

Traffic congestion causes substantial variation in travel time during different hours of a day. This significantly influences travelling decisions. In the context of supply chain, the travelling decisions with time varying travel times are triggered in a Time Dependent Location Routing Problem (TDLRP). Hence, in this paper an exact formulation of the TDLRP is presented in which the time taken to travel between each pair of nodes is a function of time. The problem formulation eliminates the waiting times at customer locations and also can tackle the problem with different scenarios such as, no time windows, with hard and soft time windows, and time dependent demand. The presented Integer Non-Linear model is linearized and solved using CPLEX. The Branch and Bound approach and other cutting approaches are used for solving the model. The results show that the pure Branch and Bound provides the results faster than cutting approaches for small size problems.

Keywords: Time Dependent Location Routing Problem, Location Routing Problem, Vehicle Routing Problem, and Vehicle Routing Problem with Time Window

*Corresponding Author. E-mail address: krishna.krishnan@wichita.edu

I. INTRODUCTION

The problem under investigation in this paper is a Location Routing Problem (LRP). LRP is the combination of two problems Location-Allocation Problem (LAP) and Vehicle Routing Problems (VRP) (Christofides and Eilon, 1969). LAP is the problem of locating a set of potential facilities and allocating customers to the locations with an objective of cost minimization (Fisher and Jaikumar, 1981). On the other hand, VRP is the problem of finding a set of routes originating from a set of depots to serve a set of customers with known demands. Each customer must be visited only once and all vehicles return to the depot from which they departed. Also, cumulative customer demands in a route should not exceed the vehicle capacity (Arntzen and Brown, 1995). Since the location of a Distribution Center (DC) impacts the routing of vehicles, LAP and VRP are investigated together in a more comprehensive problem called LRP.

There are four major problems which have to be tackled in a supply chain network - production, location-allocation, inventory, and transportation (routing). Among the four, location–allocation and routing are usually considered as the core problems of supply chain logistics (Larson and Odoni, 1981). These are usually refered to as LRP. Time Dependent LRP (TDLRP) is a variant of LRP in which the travel times between nodes in the network is not

constant and may change depending on the time at which the travel occurs. Even though LRP has been vastly investigated in literature, research on TDLRP is very scarce (Figliozzi, (2009), Ichoua, Gendreau, and Potvin, (2003)). Existing research work usually addresses only the VRP with time dependent travel times and do not approach the location and routing problem simultaneously. Orda and Rom (1990) proposed an algorithm for the shortest problem in which an arbitrary function for a link delay is allowed. The objective of the work by Orda and Rom is to find the shortest path and minimum delay under different waiting constraints. Ahn and Shin (1991) developed a heuristic for VRP with time window constraints and time-varying congestion. The heuristic is a modification of the saving, insertion, and local improvement algorithms. Hill and Benton (1992) presented a method for estimating the time dependent travel speed and a heuristic to solve the Time Dependent Vehicle Routing Problem (TDVRP). Malandraki and Daskin (1992) presented a Mixed Integer Linear Programming (MILP) model for the TDVRP with time window constraints. They developed a heuristic algorithm using the nearest neighborhood heuristic for VRP without time windows. In addition, a mathematical heuristic for the TDVRP with time windows was also developed. In this work, waiting time is allowed at nodes. The step functions considered for the travel times are symmetric. Ichoua, Gendreau, and Potvin (2003) presented a heuristic solution methodology based on Tabu-search algorithm. Figliozzi (2009) presented a flow-arc formulation for the TDVRP with hard and soft time windows along with heuristic algorithms for solving the problem. Most recent efforts regarding the TDLRP has focused on developing a heuristic/meta-heuristic for solving the problem (Hashimoto, Yagiura and Ibraka, (2008); Donati, Montemanni, Casagrande, Rizzoli, and Gmabardella, (2008); Zheng-yu, Dong-yuan, and Shang, (2010)).

All efforts in the formulation of TDVRP in literature incorporate the assumption of time windows and to the best knowledge of the authors there is no formulation for the TDVRP without consideration of time windows. Time windows define a time period during which the customer can be served. The assumption of time windows is used to simplify the formulation of the problem and to calculate the arrival time at a node according to two conditions: 1) sum of the arrival time at a customer and the travel time from the current customer to the next customer should be less than the latest arrival at the next customer; and 2) the service start time at each customer should be within a specified time window. These two conditions may sometimes lead to large and unrealistic waiting times at customer locations. For instance, consider a truck which is at a customer location at time 40. The time window for serving the next customer is [200, 260] and the travel time to the next customer is 20 minutes. Based on the traditional formulation, the arrival time at the next customer can be any time between 200 and 260 which results in a waiting time of at least 140 minutes for the truck before serving the next customer, which is not practical, especially if the time unit is large e.g. hours, days, etc. The main weakness of methods presented for TDVRP in literature can be categorized into two groups:

1. The travel time function is usually considered as a discrete step function. Due to this assumption, waiting times have to be permitted at customer location in order to get feasible solutions.

2. The effort toward applying a continuous travel time function is very rare and the formulations presented are too complicated and intricate to be analytically solved. A continuous travel function will allow improved modeling of the travel times, especially when the time intervals are for a day or shorter. This also allows easier analysis of the travel function, when the time changes from a highly congested to less congested or vice-versa.

The objectives of this paper are to alleviate these two shortcomings in the current literature and enhance the TDLRP formulation. The goals of this research are to develop a formulation for TDLRP when travel times are a function of time (discrete or continuous) and no waiting time is allowed at a customer location. The travel time function can be developed from historical data of traffic congestion. A TDLRP in which each customer`s demand has to be satisfied within a time interval is called TDLRP with Time Window (TDLRPTW). In literature, whenever the TDLRP is investigated, it is in fact a TDLRPTW and it seems that the TDLRP and time windows are not separable. In literature of TDLRP, there is no formulation of the problem in which there is no time window. The formulation of TDLRP without time window is necessary for eliminating the waiting times at customers' location as described earlier. In this research, TDLRP formulations with hard and soft time windows are also developed. The formulation is further extended to include time dependent demand. By applying the formulations proposed in this paper, it is expected that the best strategy regarding the location of DCs, allocation of customer to DCs and the routing plan from DCs to customers can be determined.

Section 2 of this paper provides a detailed definition of the problem under investigation. Section 3 is devoted to the development of the mathematical formulation for the TDLRP with different scenarios. In Section 4 an illustrative example is solved for each of the mathematical models presented in section 3. The linearization of the MINLP and the solution approach are investigated in section 5. Finally, section 6 provides conclusion and future research directions.

II. PROBLEM STATEMENT

In this section, notations used for the formulation of the problem are presented. There are N customers and M depots in the problem. The collective set of DCs and customers in the network is represented by nodes. Nodes 1 to N represent customers and nodes N+1 to N+M represent DCs. The decision variables in the formulation, provides the assignment of customers to vehicles, vehicles to DCs, as well as the sequence of visits by each vehicle. When nodes are assigned to a vehicle, a route is formed. Thus, a route is formed by a set of nodes. Position of a node in the route, is the order in which the node is visited by the vehicle. For instance, if node g is in position 2 of vehicle 1's route, it implies that node g is the second node visited by vehicle 1. D is the set of possible positions that a customer can take in a route. A vehicle may be assigned to visit at the most all N customers. Thus, the maximum number of possible positions for a customer is equal to N. The decision variables, X_{mgv} = 1, implies that node g (a customer or a DC) is the mth node visited by vehicle v. Or in other words node g is the mth node in the route assigned to vehicle v. Thus, in the definition of the notations, vehicle and route are used alternatively since they represent the same concept.

N	Total number of customers
M	Total number of DCs
K	Total number of vehicles
I	Set of customers, $I= \{1, 2,\ldots, N\}$
J	Set of DCs $J= \{N+1,2,\ldots, N+M\}$
D	Set of possible positions that a customer can take in a route, $D= \{1, 2,\ldots, N\}$
V	Set of Vehicles, $V= \{1,2,\ldots,K\}$
Y_v	Capacity of vehicle v

E_g	Customer g fulfillment level, $\forall g \in I$
g/h	Index used for all nodes
X_{mgv}	$\begin{cases} 1 \text{ if node } g \text{ is in position } m \text{ of the route } v; \\ 0 \text{ otherwise} \end{cases} \quad \forall g \in \{I \cup J\}; \forall m \in D; \forall v \in V$
P_{mv}	$\begin{cases} 1 \text{ if } m \text{ is the last taken position of route } v; \\ 0 \text{ otherwise} \end{cases} \quad \forall m \in D; \forall v \in V$
O_g	$\begin{cases} 1 \text{ if there is any vehicle assigned to node } g; \\ 0 \text{ otherwise} \end{cases} \quad \forall g \in J$
z_{vh}	$\begin{cases} 1 \text{ If vehicle } v \text{ is assigned to node h;} \\ 0 \text{ otherwise} \end{cases} \quad \forall h \in J; \forall v \in V$
C	Cost per unit of time (Labor cost, vehicle cost etc.)
q_g	Fixed cost for establishing node g, $\forall g \in J$
d_g	Initial demand of node g, $\forall g \in I$
$f_g(t)$	Demand function of node g at time t, $\forall g \in I$
S_g	Service time at node g, $\forall g \in I$
A_{mv}	Arrival time at position m of route v, $\forall m \in D; \forall v \in V$
t_{mv}	Cumulative departure time from position m on route v, $\forall m \in D; \forall v \in V$
T_{mv}	Departure time from position m on route v (between 0:00 and 24:00), $\forall m \in D; v \in V$
F_{gh} <t>	Travel time function between nodes g and h, $\forall g, h \in \{I \cup J\}$
B_g	Departure cost from Distribution Center $g, \forall g \in J$
α_{mv}	$\begin{cases} 1 \text{ if earliest arrival at position } m \text{ of route } v \text{ is violated;} \\ 0 \text{ otherwise} \end{cases} \quad \forall m \in D; \forall v \in V$
β_{mv}	$\begin{cases} 1 \text{ if latest arrival at position } m \text{ of route } v \text{ is violated;} \\ 0 \text{ otherwise} \end{cases} \quad \forall m \in D; \forall v \in V$

a_g	Earliest arrival time at customer g, $\forall g \in I$
b_g	Latest arrival time at customer g, $\forall g \in I$
Δa_g	Maximum Deviation permitted from the earliest arrival time at customer g, $\forall g \in I$
Δb_g	Maximum Deviation permitted from the latest arrival time at customer g, $\forall g \in I$
w^1_g	1 if demand has decreased at the delivery time to customer g; 0 otherwise, $\forall g \in I$
w^2_g	1 if demand has increased at the delivery time to customer g; 0 otherwise, $\forall g \in I$
λ	Lost order cost
γ	Percentage change in unit price of extra product delivered
Ω	Profit obtained from selling a unit of product
ρ_g	penalty cost associated with violation of lower time limit at customer g, $\forall g \in I$
φ_g	penalty cost associated with violation of upper time limit at customer g, $\forall g \in I$

To ensure that the vehicles' depart always from a DC, position zero of each route is reserved for a DC. Hence, only DCs can be assigned to position 0 and DCs cannot take any other position in a route.

Each customer $g \in I$ has a demand d_g which is less than the vehicle capacity, Y_v. The travel time between each pair of nodes in the system is a function of time, $F_{gh}<t>$, which is derived from historical data. There are K heterogeneous vehicles available and each vehicle departs DCs fully loaded. In addition, M locations are available for the establishment of DCs. If a DC is established, it incurs a cost of q_g, $g \in J$ in the system. There is only one type of product in the system. The objective of the problem is to find the location for establishment of DCs and routing plan for vehicles in order to minimize the system cost.

X_{mgv}'s are the decision variables defined to model the problem. X_{mgv} is a binary variable for $\forall g \in \{I \cup J\}, \forall m \in D, \forall v \in V$ which takes a value of 1, if node g (a customer or a DC) is placed in order (position) m of vehicle v's routing plan; and otherwise it is 0. There are additional terms defined to simplify the representation of the objective function and the constraints in the problem formulation. P_{mv}, presented in (1) is used to determine the final position taken on route v.

$$P_{mv} = \begin{cases} \sum_{g \in I} X_{mgv} & m = N \\ \prod_{m'=m+1}^{N} (1 - P_{m'v}) \sum_{g \in I} X_{mgv} & \forall m \in \{D / \{N\}\} \end{cases} \quad v \in V \tag{1}$$

Position 0 of each route is reserved for a DC. If the binary variable,

$X_{0gv} = 1, \forall g \in J, \forall v \in V$, it means that vehicle v is assigned to distribution center g. However, distribution center g is not opened until a customer is also assigned to the vehicle v. Thus, g will be vehicle v's *DC* only if there is a link between the depot and a customer in the network. z_{vh} defined in (2) ensures that a connection exists between a DC and a customer in the system. z_{vh} is similar to the connectivity constraint between LAP and VRP in traditional formulation of LRP, and connects the location decision to routing decisions.

$$z_{vh} = \sum_{g \in I} X_{0hv} . X_{1gv} \qquad \forall h \in J; v \in V \tag{2}$$

The value of t_{mv} is calculated through a set of recursive equations given by (3). The value of t_{mv} can be considered to be 0, if the start time is set as zero.

$$\begin{cases} t_{0v} = 0 \\ t_{mv} = \left(\sum_{m'=1}^{m} \sum_{h \in I} \sum_{g \in \{I \cup J\}} \left(F_{gh} \left\langle t_{(m-1)v} \right\rangle + S_h \right) X_{m'-1gv} X_{m'hv} \right) \left(\sum_{g \in I} X_{mgv} \right) \end{cases} \qquad \forall m \in D; \forall v \in V \tag{3}$$

If the travel time function repeats itself after H units of time, (3) must be replaced by (4) to calculate the t_{mv} $\forall m \in D; \forall v \in V$.

$$\begin{cases} T_{ov} = t_{0v} = 0 \\ t_{mv} = \left(\sum_{m'=1}^{m} \sum_{h \in I} \sum_{g \in \{I \cup J\}} \left(F_{gh} \left\langle T_{(m-1)v} \right\rangle + S_h \right) X_{m'-1gv} X_{m'hv} \right) \left(\sum_{g \in I} X_{mgv} \right) \\ T_{mv} = Mod(t_{mv}, H) \end{cases} \qquad \forall m \in D; \forall v \in V \tag{4}$$

Where, Mod function returns the reminder of dividing t_{mv} by H. For instance, if the travel time functions are represented in unit of hour and they repeat every day, the value of H will be 24. In the following section, the mathematical formulation of the problem is presented.

III. MATHEMATICAL FORMULATION OF THE PROBLEM

As already discussed one of the main drawbacks of existing formulations of TDVRP is the possibility of waiting times at customer. The formulation of the problem presented in this section overcomes this deficiency thus eliminating the waiting times at customers' location.

The TDLRP is first formulated without time window condition in section 3.1. The formulation of the TDLRP with time windows is then addressed in section 3.2 and 3.3. The formulations in section 3.2 and 3.3 ensure that the waiting time at customers' location are eliminated. The time window can be hard or soft. In a formulation with "hard time window", each customer has an associated time window during which the demand has to be met. The vehicle cannot deliver products to the customer before the start of the time window or after the time window has elapsed, i.e. late or early arrival at a customer is not acceptable (Section 3.2). In a TDLRP with "soft time window" (Section 3.3), the customer can be served before and after the preferred time window, i.e. early or late arrival at a customer is acceptable up to a predefined limit. However, when there is a late or early service to the customer, there is a penalty cost associated

with the violation of the time window. Section 3.4 presents the formulation when both demand and travel times are time dependent.

3.1 Formulation for TDLRP without time windows

The MINLP programming of the problem defined in section 2 is presented below:

Min

$$\sum_{v\in V}\sum_{m\in D} C*P_{mv}\left(t_{mv}+\sum_{h\in I}\sum_{g\in J}\left(F_{hg}\left\langle T_{mv}\right\rangle * X_{0gv} * X_{mhv}\right)\right)+\sum_{v\in V}\sum_{g\in J} z_{vg}B_g+\sum_{g\in J} q_g O_g \tag{5}$$

Subject to:

$$\sum_{v=V}\sum_{m\in D} X_{mgv}=1 \qquad \forall g\in I \tag{6}$$

$$\sum_{g\in I} X_{mgv}\le 1 \qquad \forall m\in D; v\in V \tag{7}$$

$$\sum_{g\in I}\sum_{m\in D} d_g X_{mgv}\le Y_v \qquad \forall v\in V \tag{8}$$

$$\sum_{v\in V}\sum_{g\in J}\sum_{m\in D} X_{mgv}=0 \tag{9}$$

$$\sum_{v\in V}\sum_{g\in I} X_{0gv}=0 \tag{10}$$

$$\sum_{g\in\{I\cup J\}} X_{m-1gv}\ge \sum_{g\in\{I\cup J\}} X_{mgv} \qquad \forall m\in D; v\in V \tag{11}$$

$$\sum_{g\in J} z_{vg}\le 1 \qquad \forall v\in V \tag{12}$$

$$O_g\le \sum_{v\in V} z_{vg}\le KO_g \qquad \forall g\in J \tag{13}$$

$$1\le \sum_{v\in V}\sum_{g\in J} z_{vg}\le K \tag{14}$$

$$X_{mgv}, z_{hv}, P_{mv}, O_g\in\{0,1\}$$

Equation (5) is the objective function which minimizes the total travel time, the establishment cost of DCs, and the vehicle dispatching cost from DCs. Constraint (6) ensures that each customer appears in only one route, i.e. only one route is assigned to each customer. Constraint (7) enforces that each position of a route will not be taken by more than

one customer. Constraint (8) makes sure that the total demand of customers assigned to a route is less than the vehicle capacity. It is assumed that position zero of each route is reserved for DCs. This assumption implies that DCs cannot take any other position in routes (9) and also customers cannot take position 0 of their assigned route (10). Constraint (11) ensures that position *m+1* of a route cannot be taken unless position *m* is taken. Constraint (12) ensures that a vehicle is not assigned to more than one DC. Constraint (13) determines whether a DC is open or is close. Constraint (14) keeps the total number of the vehicles between one and the number of available vehicles.

3.2 TDLRP with time windows hard time window

In this problem, it is assumed that waiting at a customer location is not allowed. The difference between the arrival and departure times at a customer is the service time. Therefore, the arrival time at a customer can be determined by (15).

$$A_{mv} = \left(t_{mv} - \sum_{m'=1}^{m} \sum_{h \in I} \sum_{g \in \{I \cup J\}} S_h X_{m'-1gv} X_{m'hv} \right) \left(\sum_{g \in I} X_{mgv} \right) \qquad \forall m \in D; \forall v \in V \tag{15}$$

Where, A_{mv} calculates the arrival time at position *m* of route *v*. When service times are zero (4) is reduced to (16).

$$A_{mv} = t_{mv} \qquad \forall m \in D; \forall v \in V \tag{16}$$

Therefore, by adding the constraint presented in (17) to the set of constraints 6-14, the model can handle LRPTD with hard time windows.

$$\sum_{g \in I} a_g X_{mgv} \le A_{mv} \le \sum_{g \in I} b_g X_{mgv} \qquad \forall m \in D; v \in V \tag{17}$$

3.3 TDLRP with Soft Time Window

As opposed to hard time windows, soft time windows allow the time interval violation with an assigned penalty cost. Therefore, the objective function (18) has a term related to the penalty cost associated with the time window violation.

Min

$$\sum_{v \in V} \sum_{m \in D} C * P_{mv} \left(t_{mv} + \sum_{h \in I} \sum_{g \in J} \left(F_{hg} \langle T_{mv} \rangle * X_{0gv} * X_{mhv} \right) \right) + \sum_{v \in V} \sum_{g \in J} z_{vg} B_g + \sum_{g \in J} q_g O_g \tag{18}$$

$$\sum_{v \in V} \sum_{m \in D} \sum_{g \in I} X_{mgv} \left(\rho_g \alpha_{mv} + \varphi_g \beta_{mv} \right)$$

The objective function minimizes the total travel time, the DCs establishment cost, vehicles dispatching cost, and the penalty costs associated with time window violation.

Time window constraint must also be modified to consider the permitted flexibility. The modified constraint is presented in (19).

$$\sum_{g\in I} a_g X_{mgv} - \left(\sum_{g\in I} \Delta a_g X_{mgv}\right)\alpha_{mv} \le A_{mv} \le \sum_{g\in I} b_g X_{mgv} + \left(\sum_{g\in I} \Delta b_g X_{mgv}\right)\beta_{mv} \qquad \forall m\in D; v\in V \tag{19}$$

$$\alpha_{mv}, \beta_{mv} \in \{0,1\}$$

Constraint 19 is used to enforce that the arrival time at each position of a route must be within the related time interval with associated allowed deviation. Thus, by replacing the objective function presented in (5) with the term presented in (18), and adding constraint (19) to the set of constraints (6-14), the model is modified to be applied to LRPTD with soft time windows.

3.4 LRP with time and demand as function of time

In conventional LRP with Time Window (LRPTW), each customer has an initial demand called D_g. The demand may either exist or expire depending on the arrival time at the customer. This statement is shown by the mathematical formulation presented in (20) (Mirzaei and Krishnan, 2011).

$$f(\tau_g) = \begin{cases} 0 & if \quad \tau_g \le a_g \\ d_g & if \quad a_g \le \tau_g \le b_g \\ 0 & if \quad \tau_g \ge b_g \end{cases} \qquad \forall g \in I \tag{20}$$

Where a_g is the earliest arrival time at node g, and b_g is the latest arrival time at node g. Also, based on the basic assumptions of vehicle routing problems, $d_g \le VC, \forall g \in I$.

However, in LRP with Time Dependent Demand, the initial demand is expected to change from the time of initiation, i.e. $f(\tau_g)$, $\forall g \in I$. In fact, the conventional LRPTW is a special case of time dependent demand problem in which the demand function is defined by (20). This section is devoted to the development of a formulation for the TDLRP in which demands and the travel time change are defined by parametric time-dependent functions. Arrival time at customer g can be calculated by (21).

$$\tau_g = \sum_{v\in V}\sum_{m\in D} X_{mgv} A_{mv}, \forall g \in I \tag{21}$$

The objective function of the TDLRP with time dependent demand minimizes the total cost of the system while maximizing the profit. When a customer demand changes with time, the customers demand at delivery time is not the same as its initial demand. Therefore, for customers with decreasing demand function, there is a "lost order cost" incurred in the system. This cost is the result of not meeting the demand completely or partially. i.e. $(d_g - f_g(t))*\lambda$, where λ is the lost order cost per unit of product. The profit is the product of total quantity delivered to the customer, the profit per unit, and the customer fulfillment level, i.e. Profit = $f_g(t)*E_g*\Omega$ in which E_g is the customer fulfillment level defined by (22).

$$E_g = \frac{f_g(\tau_g)}{d_g} \qquad \forall g \in I \tag{22}$$

The value of E_g is dynamic and depends on the time of delivery. The value of E_g for customers with a monotonously increasing demand function is greater than one, and for customers with a monotonously decreasing demand function will be less than one. The unit product price for the additional number of products delivered to

customers with increasing demand can be different from the initial price. For example, the price can be cheaper due to quantity discount. γ is the constant representing the percentage of decrease or increase in the price.

The objective function for TDLRP with time dependent is given in (23).

$$Min$$
$$\sum_{v\in V}\sum_{m\in D} C*P_{mv}\left(t_{mv}+\sum_{h\in I}\sum_{g\in J}\left(F_{hg}\left\langle T_{mv}\right\rangle * X_{0gv} * X_{mhv}\right)\right)+\sum_{v\in V}\sum_{g\in J} z_{vg}B_g+\sum_{g\in J} q_g O_g$$
$$+\sum_{g\in I}\left(w_g^1 * \lambda * (d_g - f_g(\tau_g)) \pm w_g^2 * \Omega * \gamma * (f_g(\tau_g) - d_g) - \Omega E_g f_g(\tau_g)\right) \tag{23}$$

Where, the first term determines the total transportation cost which is the product of the total travelling time and unit time cost. The second term is the vehicle dispatching cost from DCs. The third term includes the fixed DC establishment cost. The last term includes lost order cost, additional order cost/profit, and sale profit respectively. Two modifications are required for the set of constraints presented in (6-14) to handle the TDLRP with time dependent demand, as opposed to formulation in section 3.1. First, it is necessary to change the vehicle capacity constraint (8) by (24) to consider the demand variability.

$$\sum_{g\in I}\sum_{m\in D} f_g(\tau_g) * X_{mgv} \le Y_v \qquad \forall v \in V \tag{24}$$

Second, the set of constraints presented in (25) should be added to the set of constraints.

$$\begin{aligned} &w_g^1\left(d_g - f_g(\tau_g)\right) \ge 0 && \forall g \in I \\ &w_g^2\left(d_g - f_g(\tau_g)\right) < 0 && \forall g \in I \\ &w_g^1 + w_g^2 = 1 && \forall g \in I \end{aligned} \tag{25}$$

This set of constraints is used to determine whether a customer's demand at the time of delivery is higher or lower compared to the initial demand.

Since the TDLRP is the generic formulation for TDLRPTW it can handle both soft and hard time windows. Interested readers can refer to Mirzaei and Krishnan (2011) for more information.

IV. ILLUSTRATIVE EXAMPLES

4.1 LRPTD without time windows Constraints

A two layer network problem is used to illustrate the proposed mathematical model. The problem consists of 2 DCs and 4 customers. Nodes 1, 2, 3, and 4 represent the customers and nodes 5 and 6 represent potential DCs, respectively. Fig. 1 shows the travel time functions between DCs (nodes 5 & 6) and customers (nodes 1, 2, 3, & 4). Fig. 2 shows the travel time functions between each pair of customers. Although the model can handle asymmetric travel time functions, in this example it is assumed that the travel time functions are symmetric and they are repeated every 24 hours. For testing the model, different type of functions are used in this example. Appendix 1 shows the list of functions used for this case study.

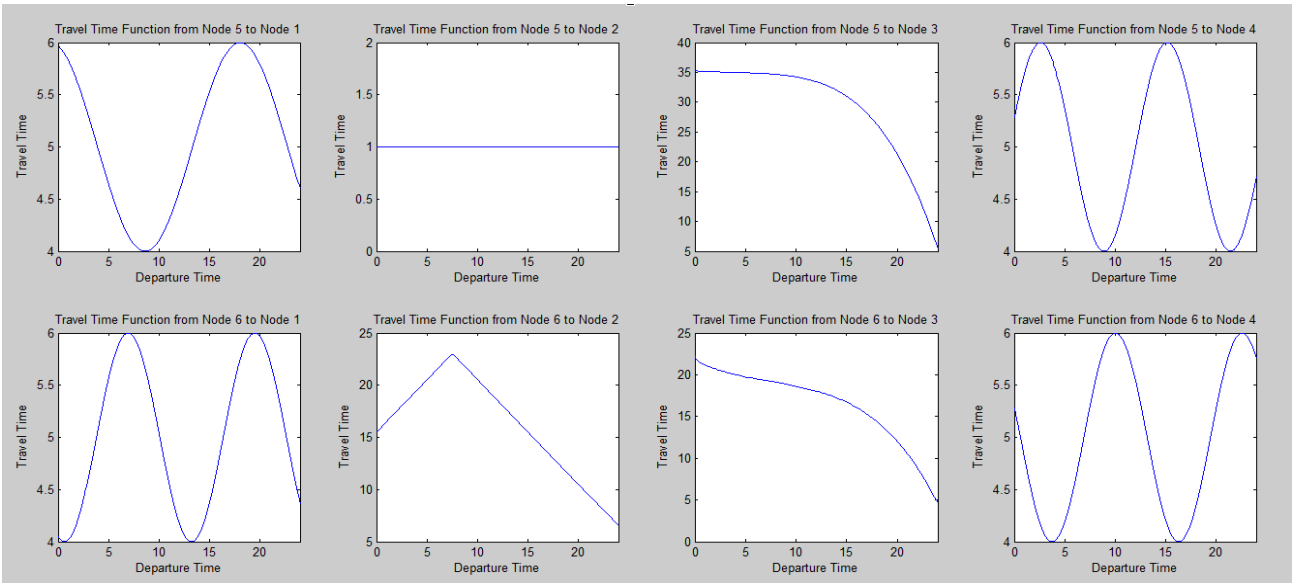

FIGURE 1. Travel Time Functions Between DCs and Customers

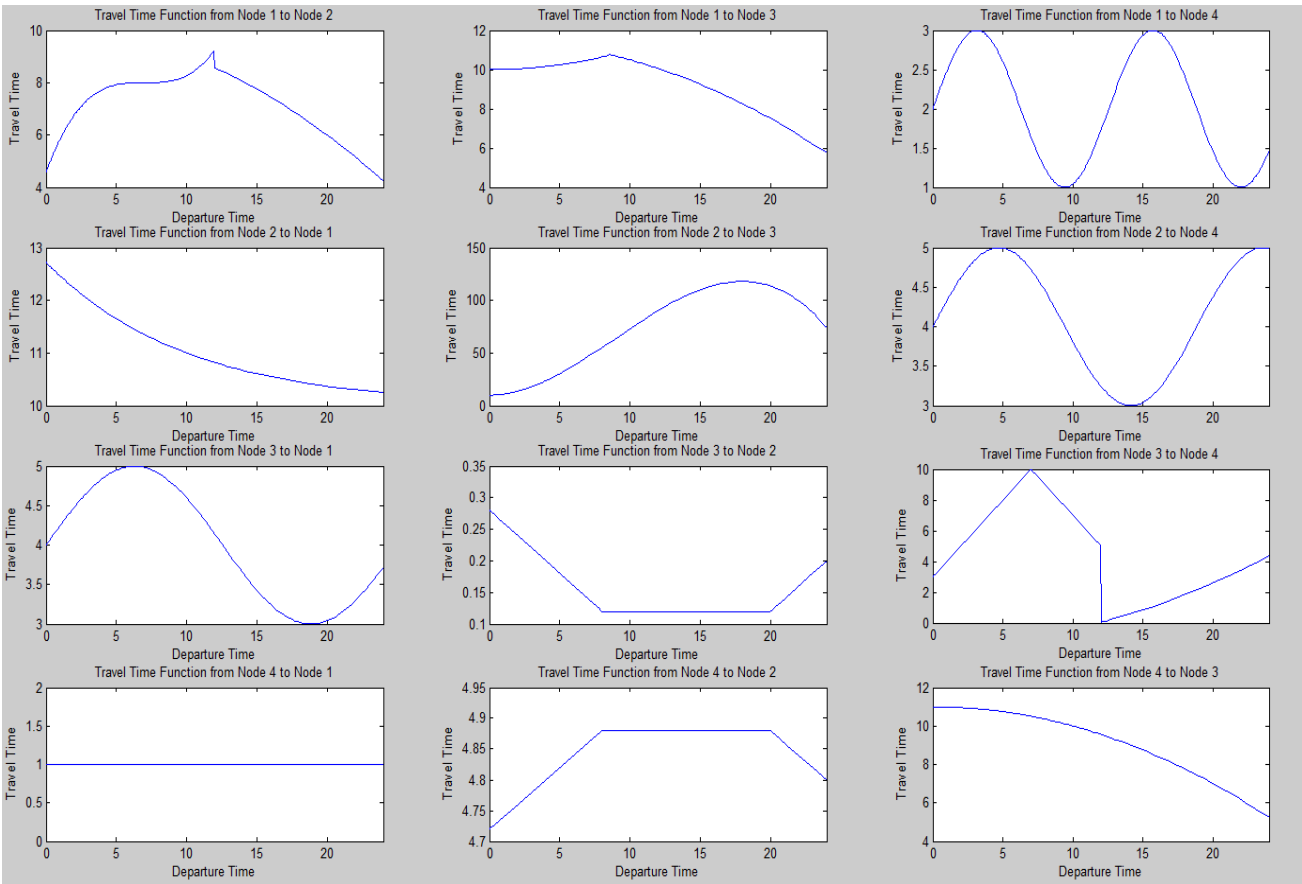

FIGURE 2. Travel Time Functions between Each Pair of Customers in the Network

There are two vehicles available with capacities of 50 and 70 units of product respectively. The departure costs from node 5 and node 6 are $45 and $50 respectively. The fixed cost of DCs in the planning horizon of the problem is $250 for node 5 and $200 for node 6.

Customers' demands for customers 1, 2, 3, and 4 are 40, 25, 20, and 10 respectively. Service times at customer's location are zero. The unit time (hour) cost for service is $3.

The mathematical formulation was solved using LINGO 12.0 optimizer software on a Pentium D CPU 3.2GHz, and 3.25 GB of RAM. The result obtained is shown in Table 1.

Table 1. RESULT OBTAINED FOR TDLRP

Objective Value	619.06
Computation Time	00:07:36
Variables with Value of 1	X_{112}, X_{131}, X_{221}, X_{242}, X_{061}, X_{062}, z_{16}, z_{26}, P_{21}, P_{22}

The result presented in the table implies that there are two routes in the network. The first route is 6-3-2-6 assigned to vehicle 1 with capacity of 50 units and the second route is 6-1-4-6 assigned to vehicle 2 with capacity of 70 units. Both routes are assigned to depot 2 (node 6).

4.2. LRPTD with Hard Time Window Constraints

The example is similar to the one presented in Section 4.1 with the difference that each customer has a time window assigned to it (Table 2). Each customer's demand is equal to its initial demand, D_g, if it is served within the specified time window; otherwise it is zero.

The mathematical formulation presented in 3.2 was solved using LINGO 12.0 optimizer software on a Pentium D CPU 3.2GHz, and 3.25 GB of RAM. The result obtained is shown in Table 3.

TABLE 2. TIME INTERVAL ASSIGNED TO EACH CUSTOMER

Customer number	1	2	3	4
Time interval	[20,27]	[14,16]	[20,25]	[15,20]

TABLE 3. RESULT OBTAINED FOR TDLRP WITH HARD TIME WINDOW

Objective Value	837.53
Computation Time	00:00:40
Variables with Value of 1	X_{121}, X_{132}, X_{212}, X_{241}, X_{061}, X_{062}, O_6, z_{16}, z_{26}, P_{21}, P_{22},

The result presented in the table implies that there are two routes in the network. The first route is 6-2-4-6 assigned to vehicle 1 with capacity of 50 units and the second route is 6-3-1-6 assigned to vehicle 2 with capacity of 70 units. Both routes are assigned to depot 2 (node 6).

4.3. LRPTD with Soft Time Window Constraints

The example in this part is similar to the example of Section 4.2 with the difference that violations from time interval are allowed up to a specified limit. The permitted violations and penalty costs associated with them are provided in Table 4. Since the problem investigated in Section 4.2 has an optimal solution with hard time window constraint, solving the same problem with soft time windows will result in no difference in terms of the final answer. Hence, to illustrate the impact of soft time window constraints and the formulation on the solution, the time windows are tightened in Table 4.

The mathematical formulation presented in Section 3.3 was solved using LINGO 12.0 optimizer software on a Pentium D CPU 3.2GHz, and 3.25 GB of RAM. The result obtained is shown in Table 5.

Although, the route configuration from Table 5 is similar to the one obtained in Section 4.2, the objective function value is larger. This is a result of tightening the time windows and allowing the model to violate the time intervals by accepting the associated penalty costs. As shown in the table, α_{11}, α_{22}, β_{12}, and β_{21} have values of one which implies that the lower limit of time windows in customers 2 and 1, and upper limit of time windows in customers 4 and 1 are violated. From Table 4, the consequence of these violations is $60 which is added to the objective value obtained in Table 3. The objective function has a value of 897.53 for the TDLRP with soft time window.

TABLE 4. TIME INTERVAL ASSIGNED TO EACH CUSTOMER

Customer number	1	2	3	4
Time interval	[26,26]	[16,17]	[20,21]	[15,18]
Upper limit violation allowed	1	0.5	1.5	1
Lower limit violation allowed	0.5	0.75	1.5	1
ρ_g ($)	20	10	50	30
φ_g ($)	50	70	20	10

TABLE 5. RESULT OBTAINED FOR TDLRP WITH SOFT TIME WINDOWS

Objective Value	897.53
Computation Time	00:01:02
Variables with Value of 1	X_{121}, X_{132}, X_{212}, X_{241}, X_{061}, X_{062}, O_6, z_{16}, z_{26}, P_{21}, P_{22}, α_{11}, α_{22}, β_{12}, β_{21}

4.4. TDLRP with Time Dependent Demand

The example is similar to the one discussed in section 4.1 with the difference that the customers' demands are not static. Each customer has a unique demand function as presented in Table 6.

TABLE 6. CUSTOMERS' DEMAND INFORMATION

	Initial Demand	Demand Function
1	40	$40+0.2\tau_1$
2	25	$25-0.2\tau_1$
3	20	$20+0.3\tau_1$
4	10	$10-0.5\tau_1$

The mathematical formulation presented in Section 3.4 was solved using LINGO 12.0 optimizer software on a Pentium D CPU 3.2GHz, and 3.25 GB of RAM. The result obtained is shown in Table 7.

TABLE 7. RESULT OBTAINED FOR TDLR

Objective Value	-386.11
Computation Time	00:10:38
Variables with Value of 1	X_{111}, X_{132}, X_{222}, X_{241}, X_{061}, X_{062}, O_6, z_{16}, z_{26}, P_{21}, P_{22}, w_2, w_4

The result presented in the table implies that there are two routes in the network. The first route is 6-1-4-6 assigned to vehicle 1 with capacity of 50 units and the second route is 6-3-2-6 assigned to vehicle 2 with capacity of 70 units. Both routes are assigned to depot 2 (node 6). The negative value of objective function shows the network profit which is $386.11.

For solving the MINLP presented in this section, LINGO optimization software was used. This software uses branch and bound method to solve the MINLP. Other commercial packages such as CPLEX gives researchers more options for choosing their problem solving method in a faster amount of time. However, CPLEX has limitations and cannot solve MINLP directly. A MINLP should be first modified to a Mixed-Integer Linear Programming (MILP) or a

quadratic formulation to be tackled by CPLEX. Hence in the next section of this paper, the model presented in Section 3.1 is first linearized and then different cutting approaches are tested to find the best solution strategy for solving the problem.

V. LINEARIZATION AND SOLUTION APPROACH

For solving a MILP there are several models developed in literature such as cutting algorithms, bender decomposition, lagrangian relaxation, etc. However, the formulation presented in section 3.1 is a MINLP and it is necessary to transform it to a MILP to be able to test other solution approaches. A closer look at the model reveals that it is in fact a constrained nonlinear binary programming and depending on the type of travel time function different approaches can be taken for the Linearization. Since, the most commonly used type of function in literature are discrete step functions, it is assumed that the travel functions are discrete step functions and the model is linearized accordingly. A general polynomial term can be presented by (26),

$$y = \prod_{j \in S} x_j \tag{26}$$

Where $S \subseteq \{1, 2, ..., n\}$ and $x_j \in \{0,1\}, j \in S$. Since all x_j are binary variables, y is a binary variable as well, and the nonlinear term presented in Equation 26 can be replaced by two inequalities according to the following theorem:

Theorem 1: Let $s = |S|$,
Equation $y = \prod_{j \in S} x_j \qquad x_j \in \{0,1\}$
where $S \subseteq \{1, 2, ..., n\}$ and $x_j \in \{0,1\}, j \in S$ holds if and only if

$$\sum_{j \in S} x_j - y \le s - 1, \tag{27}$$

$$-\sum_{j \in S} x_j + sy \le 0, \tag{28}$$

$$x_j \in \{0,1\}, j \in S, y \in \{0,1\} \tag{29}$$

***Proof*:** If any x_j is 0 then y=0. In this case, Constraint 27 is extra and redundant and constraint 28 becomes $y \le \sum_{j \in S} x_j / s < 1$ which implies y=0 by conditions presented in 29. If all x_j's are equal to 1, then y =1. In this case $y \ge 1$, which implies y=1 by condition presented in 28 and thus constraint 27 is redundant (Li and Sun, 2006).

By applying theorem 1, each nonlinear term in the objective function or the constraints can be linearized. For instance, the non-linear term $X_{m-1gv} X_{mhv}$ existing in the objective function can be replaced by a new variable Y_{mghv} by adding two constraints. Therefore:

$$Y_{mghv} = X_{m-1gv} X_{mhv} \qquad \forall m \in D; h, g \in \{I \cup J\} \tag{30}$$

$$\begin{cases} X_{m-1gv} + X_{mhv} - Y_{mghv} \le 1 \\ -X_{m-1gv} - X_{mhv} + 2Y_{mghv} \le 0 \end{cases} \qquad \forall m \in D; \forall h, g \in I \tag{31}$$

The same concept can be applied to all nonlinear terms. The additional list of variables defined for the purpose of linearization is as follows:

$$U_{m'mghv} = Y_{m'ghv} y_{mv} \qquad \forall m \in D; g,h \in I; \forall v \in V \tag{32}$$

$$\overline{P}_{mv} = 1 - P_{mv} \qquad \forall m \in \{D / \{N\}\}; \forall v \in V \tag{33}$$

$$E_{mv} = \prod_{m'=m+1}^{N} \overline{P}_{m'v} y_{mv} \rightarrow s = N - m + 1 \qquad \forall m \in \{D / \{N\}\}; \forall v \in V \tag{34}$$

$$\vartheta_{m'mghv} = E_{mv} U_{m'mghv} \qquad \forall m \in D; \{\forall m' \in D | m' <= m\}; \forall h, g \in I \tag{35}$$

$$\psi_{mghv} = E_{mv} X_{0gv} X_{mhv} \qquad \forall m \in D; h \in I; g \in J; \forall v \in V \tag{36}$$

$\sum_{g \in I} X_{mgv}$ shows whether there is a customer assigned to position *m* of route or not. Hence, it works like a binary variable. Thus, for simplifying the linearization of the problem $\sum_{g \in I} X_{mgv}$ is also replaced by a binary variable presented in 37.

$$\sum_{g \in I} X_{mgv} - y_{mv} = 0 \qquad y_{mv} \in \{0,1\} \tag{37}$$

By using the new variables defined, the linearized model is presented below:

Min

$$\sum_{v \in V} \sum_{m \in D} C \left(\sum_{m'=1}^{m} \sum_{h \in I} \sum_{g \in I} \left(F_{gh} \langle t_{(m-1)v} \rangle + S_h \right) \vartheta_{m'mghv} + \sum_{h \in I} \sum_{g \in J} \left(F_{hg} \langle t_{mv} \rangle \psi_{mhgv} \right) \right) + \tag{38}$$

$$\sum_{v \in V} \sum_{g \in J} \sum_{h \in I} Y_{1ghv} B_g + \sum_{g \in J} q_g O_g$$

$$\sum_{v=V} \sum_{m \in D} X_{mgv} = 1 \qquad \forall g \in I \tag{39}$$

$$\sum_{g \in I} X_{mgv} \leq 1 \qquad \forall m \in I; v \in V \tag{40}$$

$$\sum_{g\in I}\sum_{m\in D} d_g X_{mgv} \le Y_v \qquad \forall v \in V \tag{41}$$

$$\sum_{v\in V}\sum_{g\in J}\sum_{m\in D} X_{mgv} = 0 \tag{42}$$

$$\sum_{v\in V}\sum_{g\in I} X_{0gv} = 0 \tag{43}$$

$$\sum_{g\in\{I\cup J\}} X_{m-1gv} \ge \sum_{g\in\{I\cup J\}} X_{mgv} \qquad \forall m \in D; v \in V \tag{44}$$

$$\sum_{g\in J}\sum_{h\in I} Y_{1ghv} \le 1 \qquad \forall v \in V \tag{45}$$

$$O_g \le \sum_{v\in V}\sum_{h\in I} Y_{1ghv} \le KO_g \qquad \forall g \in J \tag{46}$$

$$1 \le \sum_{v\in V}\sum_{g\in J}\sum_{h\in I} Y_{1ghv} \le K \tag{47}$$

$$\begin{cases} X_{m-1gv} + X_{mhv} - Y_{mghv} \le 1 \\ -X_{m-1gv} - X_{mhv} + 2Y_{mghv} \le 0 \end{cases} \qquad \forall m \in D; \forall h, g \in I \tag{48}$$

$$\begin{cases} Y_{m'ghv} + y_{mv} - U_{m'mghv} \le 1 \\ -Y_{m'ghv} - y_{mv} + 2U_{m'mghv} \le 0 \end{cases} \qquad \forall m \in D; \left\{\forall m' \in D \middle| m' \le m\right\}; \forall h, g \in I \tag{49}$$

$$\begin{cases} \sum_{m'=m+1}^{N} \overline{P}_{m'v} + y_{mv} - E_{mv} \le N - m \\ -\sum_{m'=m+1}^{N} \overline{P}_{m'v} - y_{mv} + (N-m+1)E_{mv} \le 0 \end{cases} \qquad \forall m \in D; \left\{\forall m' \in D \middle| m' \le m\right\}; \forall h, g \in I \tag{50}$$

$$\begin{cases} E_{mv} + U_{m'mghv} - \vartheta_{m'mghv} \le 1 \\ -E_{mv} - U_{m'mghv} + 2\vartheta_{m'mghv} \le 0 \end{cases} \qquad \forall m \in D; \left\{\forall m' \in D \middle| m' \le m\right\}; \forall h, g \in I \tag{51}$$

$$\begin{cases} E_{mv} + X_{0gv} + X_{mhv} - \psi_{mghv} \le 2 \\ -E_{mv} - X_{0gv} - X_{mhv} + 3\psi_{mghv} \le 0 \end{cases} \qquad \forall m \in D, \left\{\forall m' \in D \middle| m' \le m\right\}; \forall h, g \in I \tag{52}$$

$$X_{mgv}, Y_{mghv}, \vartheta_{m'mghv}, \psi_{mghv}, U_{m'mghv} \in \{0,1\}$$

Where, constraints (48-52) are added to the model for the purpose of linearization. The linearized model is programmed in CPLEX for 4 different examples and the data for the examples are provided in Table 8. The examples are then solved with multiple solution strategies such as branch and bound, Clique cuts, GUB Cover cuts, Implied Bound Cuts, Gomory Fractional cuts, and Zero-half cuts to find the fastest method for solving the problem.

TABLE 8. INFORMATION REGARDING THE EXAMPLES SOLVED

Example	# of customers	# of DCs	# of vehicles	Vehicles Capacity
1	4	2	2	50,70
2	5	2	2	50,70
3	6	2	2	50,70
4	7	2	3	50,70,70

Tables 9 to 12 show the number of cuts, and computation time for each solution approach. In the first row of the tables, value of -1 implies that no cutting strategy is used; 0 implies that the automatic setting of CPLEX for applying cuts is used; 1 implies that cuts are used moderately; and 2 implies the aggressive usage of cuts in generating the results. As reflected in the tables, pure branch and bound has the smallest computational time. Therefore, in case the problem sizes are small and computational times are not an issue, there is no need to linearize the model and Lingo can be an appropriate interface to solve the model. This conclusion is only true when the problem is not large and the travel times are discrete step functions. The evaluation of solution approaches for different types of travel time functions, demand functions, and problem size is considered as future research.

TABLE 9. REULSTS OBTAINED FOR EXAMPLE #1

6 node example -Obj. function value= 5.350×10^2				
	-1	0	1	2
Clique cuts	0	3	2	4
GUB Cover cuts	0	4		
Implied Bound Cuts	0	62	4	5
Gomory Fractional cuts	0	2	2	0
Zero-half cuts	0	0	0	1
Solution time	0.19	0.34	0.30	0.67

TABLE 10. RESULT OBTAINED FOR EXAMPLE#2

7 node example, Obj. function value= 2.675×10^2				
Cut setting	-1	0	1	2
Clique cuts	0	2	2	2
GUB Cover cuts	0	2	0	2
Implied Bound Cuts	0	2	2	2
Gomory Fractional cuts	0	2	2	4
Zero-half cuts	0	6	6	8
Solution time	0.34	1.05	0.83	1.42

TABLE 11. RESULT OBTAINED FOR EXAMPLE#3

8 node example , Obj. function value= 2.675×10^2				
Cut setting	-1	0	1	2
Clique cuts	0	0	0	1
GUB Cover cuts	0	1	1	3
Implied Bound Cuts	0	3	3	11
Gomory Fractional cuts	0	1	1	3
Zero-half cuts	0	0	0	1
Solution time	1.97	2.75	2.71	12.47

TABLE 12. RESULT OBTAINED FOR EXAMPLE#4

9 node example-Obj. function value= 2.675×10^2				
Cut setting	-1	0	1	2
Cover cuts	0	2	2	2
Implied Bound Cuts	0	5	5	5
Flow cuts	0	1	0	0
Gomory Fractional cuts	0	3	4	0
Zero-half cuts	0	1	1	0
Solution time	2.98	4.48	4.51	6.08

VI. CONCLUSION

Traffic congestion is a normal phenomenon especially in urban areas. Traffic rush hours in the morning and evening typically result in higher travel times. Thus, traffic congestion influences the time taken to travel. The main drawback of existing formulations of TDLRP is that the waiting time at customers' location is used to take care of time window violations. In the formulation presented in this paper, the assumption of time windows and waiting time at customers' location are relaxed and a step-by-step formulation of TDLRP for several scenarios has been developed. In the initial formulation, the time windows are eliminated and a TDLRP formulation that eliminates the waiting time at customer locations is first developed. The formulation was then extended to include both hard and soft time windows. The model is further enhanced to address the issue of demand variation, - i.e. assuming that the demand is a function of time. Each of the formulations is illustrated with the use of an example.

For solving the MINLP presented in section 3.1 using CPLEX, depending on the type of travel time function, different methods can be applied. Since the discrete step function is the most common representation for travel time function in literature, it is assumed that the travel time functions are discrete step functions. By assuming the discrete step functions to represent the travel times, the MINLP can be linearized for faster solution in CPLEX. For solving the MILP obtained, different solution strategies are applied including Branch and Bound, moderate and aggressive cut generators, etc. The results obtained show that when the travel time is a discrete function, the pure branch and bound method can provide results faster than cutting methods. However, as already discussed, depending on the type of travel time function, different solution methods can yield different computational time, and this is a subject of future research. In addition, since the problem is NP-hard, heuristic or meta-heuristic algorithms for solving large problems are required. A closer look at constraints 42 and 43 reveal that they can be decoupled from route/vehicle, and hence column generation or bender decomposition for finding the exact solution of larger size problems is possible and can be further investigated.

VII. REFERENCES

Ahn, B.H. and Shin, J.Y. "Vehicle-routeing with time windows and time-varying congestion", *Journal of the Operational Research Society*, 42 (5),1991, 393-400.

Arntzen, B., Brown, G., Harrison, T. and Trafton, L. "Global supply chain management at Digital Equipment Corporation". *Interfaces* 25 (1),1995, 69-93

Christofides, N. and Eilon, S. "An algorithm for the vehicle-dispatching problem", *OR*, 20 (3), 1969, 309-318.

Donati, A., Montemanni R., Casagrande, N, Rizzoli, A. E. and Gmabardella L. M., "Time dependent vehicle routing problem with a multi ant colony system", European Journal of Operational Research 185(3), 2008, 1174-1191.

Figliozzi, M.A., 'A route improvement algorithm for the vehicle routing problem with time dependent travel times', *Proceeding of the 88th Transportation Research Board Annual Meeting CD rom*, Washington DC, USA, January 2009.

Fisher, M.L. and Jaikumar, R., "A generalized assignment heuristic for vehicle routing". *Networks*, 11 (2), 1981, 109-124.

Hashimoto, H., Yagiura, M. and Ibraka, T., "An iterated local search algorithm for the time-dependent vehicle routing problem with time windows", *Discrete Optimization*,5(2), 2008, 434-456.

Hill, A.V. and Benton, W., "Modelling intra-city time dependent travel speeds for vehicle scheduling problems", *Journal of the Operational Research Society* 43 (4), 1992, 343-351,

Ichoua, S., Gendreau, M. and Potvin, J., "Vehicle dispatching with time dependent travel times", *European Journal of Operational Research*,144 (2), 2003, 379-396.

Larson, R. and Odoni, A., *Urban operations research*, Prentice Hall, Englewoods Cliffs, NJ, 1981.

Li, D. and Sun, X., *Nonlinear integer programming*, Springer, New York, 2006.

Malandraki, C. and Daskin, M., "Time dependent vehicle routing problems: Formulations, properties and heuristic algorithms", *Transportation Science*, 26 (3), 1992, 185.

Malandraki, C. and Dial, R.B., "A restricted dynamic programming heuristic algorithm for the time dependent traveling salesman problem", *European Journal of Operational Research*, 90 (1), 1996, 45-55.

Mirzaei, S. and Krishnan, K. K., 'Supply Chain Network Configuration with Time Dependent Demand', *Proceeding of Institute of Industrial Engineering Conference CD rom*, Reno, Nevada, 2011.

Orda, A. and Rom, R., "Shortest-path and minimum-delay algorithms in networks with time dependent edge-length". *Journal of the ACM (JACM)*, 37 (3), 1990, 607-625

Zheng-yu, D., Y. Dong-yuan, Y., and Shang W., "An improved genetic algorithm for Time Dependent Vehicle Routing Problem", *Proceeding of 2nd international Confernce on Future Computer and Communication (ICFCC)*,2010.

APPENDIX

Travel time functions

$$F_{12}\langle t\rangle = \begin{cases} \dfrac{(t-7)^3}{100} + 8 & 0 < t < 12 \\ \dfrac{-t}{100} + 10 & 12 \le t < 24 \end{cases}$$

$$F_{13}\langle t\rangle = \begin{cases} \dfrac{t^2}{100} + 10 & 0 < t < 8.5 \\ \dfrac{-t^2}{100} + 11.5 & 8.5 \le t < 24 \end{cases}$$

$$F_{14}\langle t\rangle = \sin(\frac{t}{2}) + 2 \qquad 0 < t < 24$$

$$F_{21}\langle t\rangle = e^{\frac{10-t}{10}} + 10 \qquad 0 < t < 24$$

$$F_{23}\langle t\rangle = t^2 - (\frac{t}{3})^3 + 10$$

$$F_{51}\langle t\rangle = \cos(\frac{x}{3} - 50) + 5 \qquad 0 < t < 24$$

$$F_{52}\langle t\rangle = 1 \qquad 0 < t < 24$$

$$F_{53}\langle t\rangle = \left(\frac{t-5}{10}\right)^2 - \left(\frac{x}{10}\right)^4 + 35 \qquad 0 < t < 24$$

$$F_{54}\langle t\rangle = \sin(\frac{t}{2} - 6) + 5 \qquad 0 < t < 24$$