# Sensitivity of Large Scale Facility Location Solutions

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In this paper we investigate the question: by how much does the solution change when different objectives are used? When demand is uniformly distributed over a large area, we show that the solution of locating multiple facilities in the area is the same for many location objectives. However, when demand is not uniformly distributed, we show in two case studies that the location for minimizing the weighted sum of distances may be quite far from the location that minimizes the maximum distance. One case study is based on the 40 largest metropolitan areas of the world, and the second case study is based on the 477 cities in the State of California. In both case studies the solution to minimizing the average distance is far away from the solution of minimizing the maximum distance. For the second case study we first show that California can be approximated well by a projection on a plane which is not the case for larger areas on a globe. Minimizing the sum of weighted distances by locating several facilities tends to yield locations near large metropolitan areas in the western part of the state while locations that minimize the maximum distance are in rural areas in the eastern part of the state.

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## I. INTRODUCTION

Location models deal with the location of one or more facilities to provide service to a set of demand points, such as customers, neighborhoods, cities. A list of $n$ demand points located at $x_i$ for $i = 1 \ldots n$ each with an associated weight $w_i$ are given. A facility needs to be located at a location $X$. The distance between the facility and demand point $i$ is $d_i(X)$. The two basic single facility location models, that are widely used, are the Weber problem (Weber, 1909) and the minimax problem (Sylvester, 1857, 1860).

The Weber problem dates back to Fermat in the 1600s. For a historical review of the Weber problem see Church (2019, Drezner et al. (2002), or Wesolowsky (1993). In his book, Weber (1909) considered the problem of locating a distribution center that minimizes the total travel distance for delivery of supplies. The Weber objective to be minimized is:

$$\sum_{i=1}^{n} w_i \, d_i(X) \qquad (1)$$

While the total travel distance is minimized in the Weber model, the distance to the farthest demand point is minimized in the minimax model. A historical review of the minimax problem is available in Drezner (2011). The minimax objective is to minimize:

$$\max_{1 \le i \le n}\{ d_i(X)\} \qquad (2)$$

Common applications of the minimax model, that are described in the literature, are locating emergency facilities such as ambulances, fire stations, police stations, etc. The objective is that the farthest customer will get the best available service. An interesting new application is designing a hub for delivery between cities following the original model of Federal Express. To provide overnight delivery between 40 cities, the company needs 1560 airplanes. However, we can select a hub where airplanes from each city arrive at about the same time, exchange packages, and then fly back to their originating city. This arrangement requires only 40 airplanes. The appropriate objective for this arrangement is to find a hub that has the minimum possible distance to the farthest city. This allows for the latest possible departure from the farthest city and the earliest possible arrival with the deliveries to every city.

The Weber problem is also called the 1-median problem because when demand points are located on a line, the solution is the median point. The minimax problem is also called the 1-center problem because the best location for the facility is the center of the smallest circle that encloses all demand points.

These problems are considered in many environments such as on a plane using Euclidean distances (Weiszfeld, 1936) ($\ell_2$), Manhattan distances (Love et al., 1988) ($\ell_1$), general $\ell_p$ distances (Brimberg and Love, 1993; Love and Morris, 1972), on a network traveling along network links (Hakimi, 1964, 1965), and on a globe using great circle distances (Drezner and Wesolowsky, 1978; Hansen et al., 1995; Katz and Cooper, 1980; Suzuki, 2019).

These problems are generalized to locating $p \geq 1$ facilities so that each demand point receives services from the closest facility. The Weber problem is generalized to the *p*-median problem (Daskin, 1995; Daskin and Maass, 2015; Kariv and Hakimi, 1979b) also termed the multi-source Weber problem (Brimberg et al., 2000; Kuenne and Soland, 1972), and the minimax problem is generalized to the *p*-center problem (Calik et al., 2015; Drezner, 1984; Kariv and Hakimi, 1979a).

Most location papers use randomly generated instances in order to test proposed solution methods (e.g. Beasley, 1990). It is common to generate random distribution of demand points in a square by a uniform distribution. We investigate the question: by how much does the solution change when different objectives are used? To the best of our knowledge no one investigated the sensitivity of the location solutions to the objective function and the distribution of the demand points. We investigate locating many facilities in a large area when demand is uniform, and contrast it with non-uniform demand in a large area. We draw different conclusions from these cases.

In Section 2 we detail location of many facilities with continuously uniform demand in a large area. There are many objectives that were investigated in this setting and it is interesting that the solutions to such problems are the same regardless of the objective used. In contrast, we investigated two cases of non-uniform demand in a large area. In Section 3 we solve the 1-median and 1-center problems on a globe where demand is generated in the 40 largest metropolitan areas. We show that the 1-median solution is located 3,853 miles from the 1-center solution. In Section 4 we solve median and center problems with demand generated at 477 California cities. We first investigate whether the Earth's curvature affects the solution. It was found that for an area of the size of California the effect is minimal. California has 3 large metropolitan areas (Los Angeles, San Francisco, and San Diego) near the Pacific Ocean. The median solutions are near these metropolitan areas but the center solutions are inland far from the large metropolitan areas.

## II. LOCATING A LARGE NUMBER OF FACILITIES

The *p*-median objective is to minimize the total distance traveled by customers to the closest facility while the *p*-center objective is to minimize the distance to the customer who is farthest from the closest facility. The *p*-center is interpreted as providing the best possible service to the worst served customer. Since each demand point is served by its closest facility, each facility serves the demand points in a polygon surrounding it. The delineation of the areas covered by the facilities are convex polygons which is termed a Voronoi diagram generated by the facilities (Okabe et al., 2000; Suzuki and Okabe, 1995; Voronoi, 1908), See, for example, the polygons in Figure 1. We show in the figure the only available symmetric patterns in the plane: square, hexagonal, and triangular. Each facility has the same pattern surrounding it.

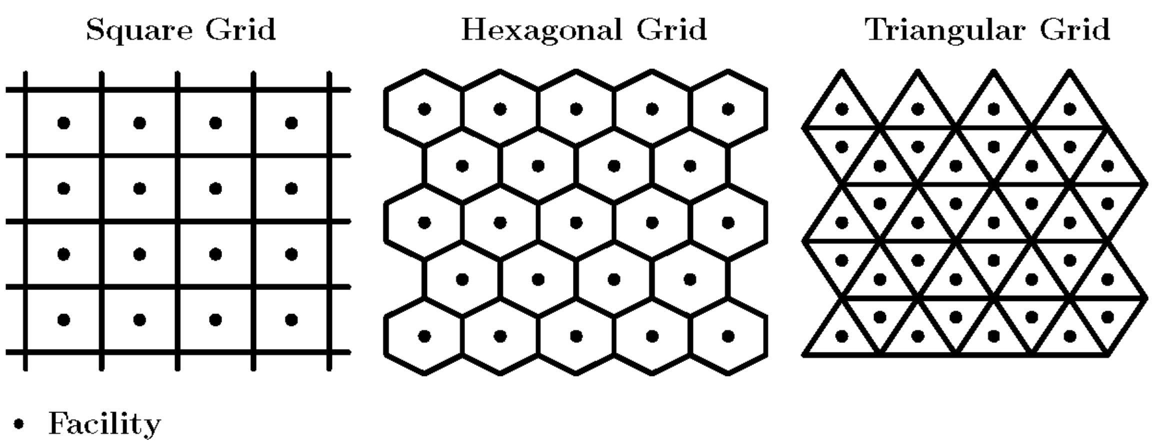


**FIGURE 1: SYMMETRIC GRIDS**

We found that the hexagonal pattern provides the best solution for many objectives when demand is uniformly distributed in a large area. For example, Hilbert and Cohn-Vossen (1932) showed that the densest non-overlapping circle configuration in the plane is with circles whose centers are in an hexagonal pattern. Drezner and Zemel (1992) showed that the best pattern for a chain of competing facilities to protect against a future competitor is hexagonal. In an hexagonal grid, a future competitor can capture up to 51.27% of the market share of an existing facility. In a square grid a competitor can capture up to 56.25% of the market share, and in a triangular grid it can capture 2/3 of it. Szabo et al. (2007) showed that the best pattern of circles' centers of packing circles in an area, such as a square area, is a hexagonal pattern. Okabe and Suzuki (1997) found that *p*-median solutions in an area are arranged in a hexagonal pattern. Suzuki and Drezner (1996) found that *p*-center solutions in a large area with uniform demand tend to be arranged in a hexagonal pattern. The hexagonal packing is found in nature as well, such as bee-hives.

If demand is not uniformly distributed, it is reasonable to expect that there will be more facilities near large population concentrations in the $p$-median objective while the $p$-center locations will be more uniformly distributed. This is demonstrated in the next two sections on real data sets.

## III. FIRST CASE STUDY: LOCATION ON THE SPHERE

The radius of the Earth is $R$ = 3959 miles. Drezner and Wesolowsky (1978) proved that if all demand points are inside a circle with a spherical distance $\frac{\pi}{4}R$ =3109 miles, all distances between points inside the circle are convex and thus the Weber problem is convex with only one local minimum which is the global one.

The complete data set of metropolitan areas in the world is available for the year 2016 in **https://simplemaps.com/data/world-cities**. The 40 largest metropolitan areas are listed in Table 1 and depicted in Figure 2. Hoornweg and Pope (2017) provide in their Table 4 forecasted values for the 21st century and one can extract their predicted data for the year 2025, or even a later year, if future values are of interest.

These 40 metropolitan areas cannot be approximated by planar distances because the distances are too large for such a transformation. For example, the shortest spherical distance between Seoul and Los Angeles passes near the North Pole and cannot be approximated by a straight line. The largest possible distance between two points on the globe (which are antipodes to one another) is $\pi R$ = 12,438 miles. The largest distance between any two of the 40 cities is 12,312 miles between Jakarta and Bogota. These two cities are very close to being antipodes.

Formulas for the great circle distances between points are given in the appendix. The smallest circle surrounding all metropolitan areas has a radius of 6,692 miles, which is obtained for three cities: Lima, Jakarta, and Los Angeles. Note that the smallest possible distance must be at least half the distance between the two farthest cities from one another (Jakarta and Bogota) of 6,156 miles. The center of this circle, which is the 1-center solution, is at 35.45°N, 12.81°E and is the best hub location for delivering packages between these 40 metropolitan areas. This location is north of the farthest three cities. However, moving a bit southward to be closer to the farthest three cities does not work. The great circle shortest distance between the center and Los Angeles passes near the North Pole so moving the center location southward increases the distance to Los Angeles even though the center's latitude is north of Los Angeles' latitude.

## TABLE 1: LOCATION AND POPULATION (IN MILLIONS OF RESIDENTS) OF THE LARGEST 40 METROPOLITAN AREAS IN THE WORLD

| Metro | Latitude† | Longitude‡ | Population | Metro | Latitude† | Longitude‡ | Population |
|---|---|---|---|---|---|---|---|
| Tokyo | 35.69 | 139.75 | 35.7 | Seoul | 37.57 | 127.00 | 9.8 |
| New York | 40.69 | -73.92 | 19.2 | Lagos | 6.44 | 3.39 | 9.5 |
| Mexico City | 19.44 | -99.13 | 19.0 | Jakarta | -6.17 | 106.83 | 9.1 |
| Mumbai | 19.02 | 72.86 | 19.0 | Guangzhou | 23.15 | 113.33 | 8.8 |
| São Paulo | -23.56 | -46.63 | 18.8 | Chicago | 41.84 | -87.69 | 8.6 |
| Delhi | 28.67 | 77.23 | 15.9 | London | 51.50 | -0.12 | 8.6 |
| Shanghai | 31.22 | 121.44 | 15.0 | Lima | -12.05 | -77.05 | 8.0 |
| Kolkata | 22.50 | 88.32 | 14.8 | Tehran | 35.67 | 51.42 | 7.9 |
| Dhaka | 23.72 | 90.41 | 12.8 | Kinshasa | -4.33 | 15.32 | 7.8 |
| Buenos Aires | -34.60 | -58.40 | 12.8 | Bogota | 4.60 | -74.08 | 7.8 |
| Los Angeles | 34.11 | -118.41 | 12.7 | Shenzhen | 22.55 | 114.12 | 7.6 |
| Karachi | 24.87 | 66.99 | 12.1 | Wuhan | 30.58 | 114.27 | 7.2 |
| Cairo | 30.05 | 31.25 | 11.9 | Hong Kong | 22.31 | 114.19 | 7.2 |
| Rio de Janeiro | -22.93 | -43.23 | 11.7 | Tianjin | 39.13 | 117.20 | 7.2 |
| Ōsaka | 34.75 | 135.46 | 11.3 | Chennai | 13.09 | 80.28 | 7.2 |
| Beijing | 39.93 | 116.39 | 11.1 | Taipei | 25.04 | 121.57 | 6.9 |
| Manila | 14.60 | 120.98 | 11.1 | Bangalore | 12.97 | 77.56 | 6.8 |
| Moscow | 55.75 | 37.62 | 10.5 | Bangkok | 13.75 | 100.52 | 6.7 |
| Istanbul | 41.11 | 29.01 | 10.1 | Lahore | 31.56 | 74.35 | 6.6 |
| Paris | 48.87 | 2.33 | 9.9 | Chongqing | 29.57 | 106.60 | 6.5 |

† Positive latitude: N. Negative latitude: S.
‡ Positive longitude: E. Negative longitude: W.


**FIGURE 2: 1-MEDIAN AND 1-CENTER AMONG 40 METROPOLITAN AREAS.**


**FIGURE 3: CLOSE-UP OF 1-MEDIAN AND 1-CENTER SOLUTIONS**

The 1-center location is in the Mediterranean near Tunisia (see Figure 3). The problem is not convex, therefore there may be several local optima. The solution was found by the global optimization method in Drezner and Wesolowsky (1983). It can also be found by the global optimization approaches in Hansen et al. (1995); Suzuki (2019). These approaches can also be used for optimally solving the 1-median problem as well as many other location models.

Hansen et al. (1995) partitioned the sphere's surface to regions which are similar to rectangles bounded by two latitudes and two longitudes. Lower and upper bounds are found in each region. If the lower bound is below the best found solution so far, the region is partitioned into four regions by central lines of latitude and longitude. If the lower bound is within a required accuracy, the region is eliminated from further search. Suzuki (2019) designed a similar approach but partitioned the surface of the sphere to triangles using a method similar to the Delaunay triangulation (Lee and Schachter, 1980) in the plane.

A list of cities, which are more than 5,000 miles from the 1-center solution, sorted by distances are given in the following table.

| | | | |
|---|---|---|---|
| Jakarta | 6692 | Seoul | 5866 |

| | | | |
|---|---|---|---|
| Lima | 6692 | Bogota | 5860 |
| Los Angeles | 6692 | Guangzhou | 5857 |
| Buenos Aires | 6669 | São Paulo | 5631 |
| Manila | 6614 | Wuhan | 5600 |
| Mexico City | 6591 | Bangkok | 5545 |
| Tokyo | 6453 | Rio de Janeiro | 5449 |
| Ōsaka | 6339 | Tianjin | 5385 |
| Taipei | 6187 | Beijing | 5318 |
| Hong Kong | 5936 | Chongqing | 5262 |
| Shenzhen | 5922 | Chicago | 5110 |
| Shanghai | 5909 | -- | -- |

The 1-median solution is at 38.11°N, 84.19°E (see Figure 3). This location is in Xinjiang Province, China, 3,853 miles from the 1-center solution. Buenos Aires is the farthest city from the 1-median location at a distance of 10,356 miles.

## IV. SECOND CASE STUDY: THE STATE OF CALIFORNIA

There are three major metropolitan areas in California: Los Angeles, San Francisco, and San Diego. Orange County is associated with Los Angeles and Silicon Valley associated with San Francisco. Many smaller communities are found throughout the state.

The website **https://simplemaps.com/data/us-cities** has data for more than 36,000 U.S. municipalities. We extracted data for latitude, longitude and population for 477 California cities, a total population of 32,423,799 people, accounting for every California resident.

We first investigated whether assuming that California is located on a plane rather than on a globe distorts the results. We projected each city to a plane that is tangent at the center of the state and converted the latitude and longitude of every point to coordinates in miles. The radius of the Earth is $R$ = 3959 miles. The number of miles separating one degree of latitude is $\frac{2\pi}{360}R \approx$ 69.098 miles. The average latitude of the 477 cities is φ = 36.1041501. The number of miles separating one degree of longitude is multiplied by cos φ yielding about 55.827 miles. The western-most city is assigned $x$=0 and the southern-most city is assigned $y$ = 0.

Drezner and Wesolowsky (1978) proved that if all demand points are inside a circle with a spherical distance $\frac{\pi}{4}R$ = 3,109 miles, all distances between points inside the circle are convex and thus the Weber problem is convex with only one local minimum which is the global one. California clearly satisfies this condition. Both the 1-median and 1-center problems on the globe and on a plane can be solved by Solver in Microsoft Excel.

The planar 1-median solution is between Los Angeles and Burbank about 3 miles from each. The planar 1-center solution is midway between Crescent City in the north and Calexico to the south at a distance of 396.86 miles from each, about 21 miles northeast of Madera. The correct spherical solution for the 1-median is close to the planar solution. For the 1-center spherical solution, the distance is 394.25 which is only 2.61 miles shorter. These results confirm that the planar approximation is usable for datasets of areas as large as California.

The 3-median solutions are depicted in Figure 5. The best 3-median locations are to locate three facilities near the large metropolitan areas, one near San Diego, one near Los Angeles, and one near San Francisco. The best locations for the 3-center problems are inland to serve inland communities as well as possible. The three locations are close to the line connecting Crescent City and Calexico with the maximum distance between demand points and their nearest facility being 167.42 miles. Thus every city is within 167.42 miles of its closest facility.

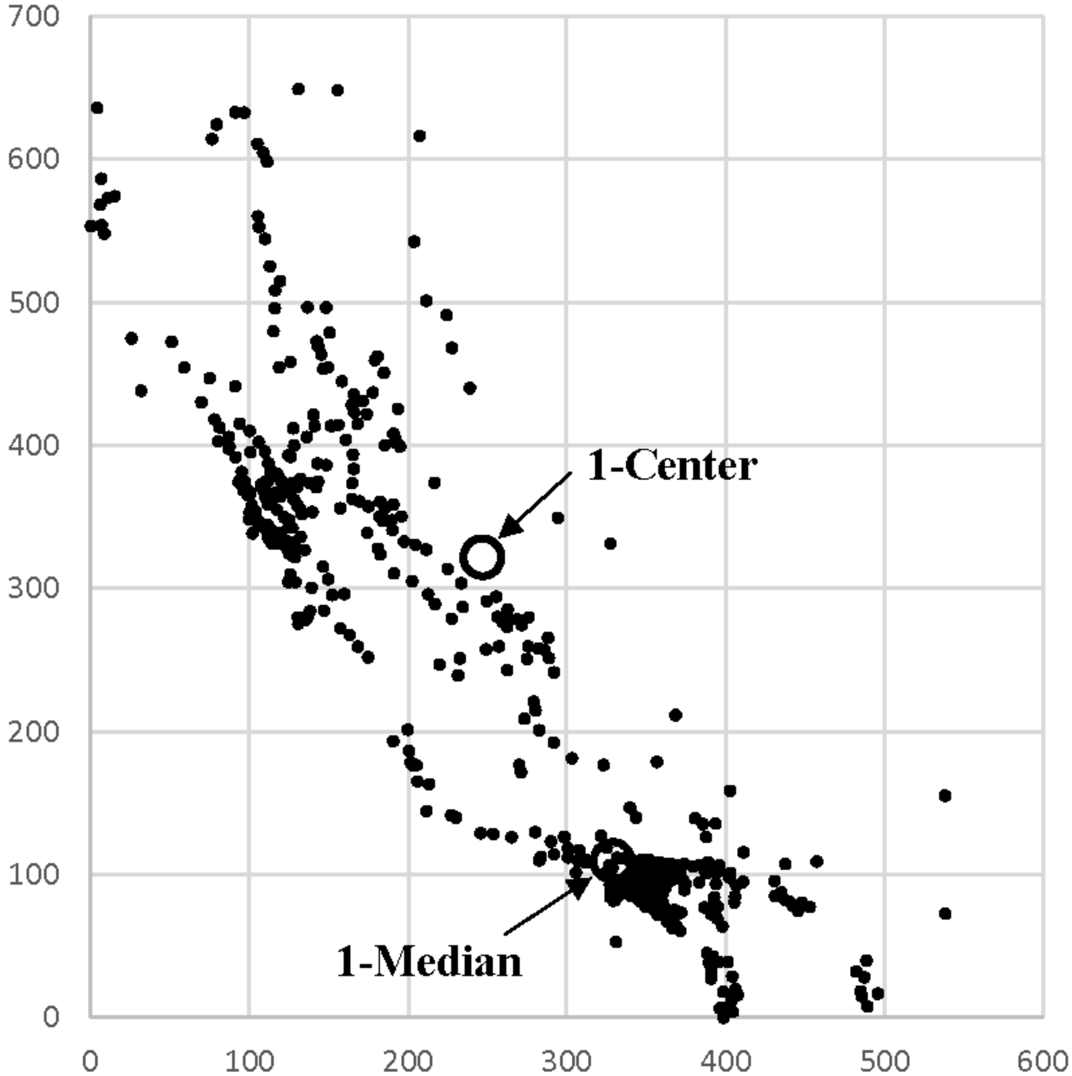


**FIGURE 4: CALIFORNIA CITIES (PLANAR PROJECTION)**

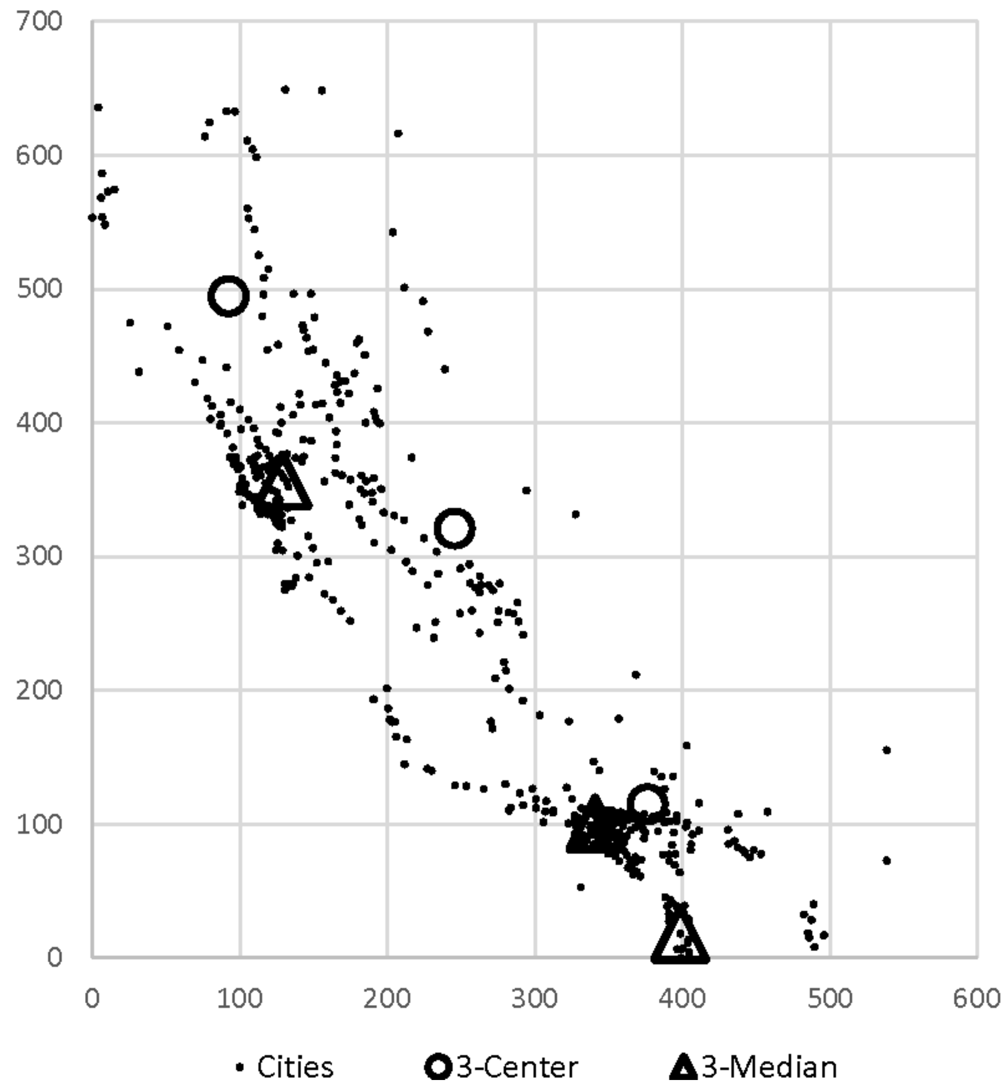


**FIGURE 5: 3-MEDIAN AND 3-CENTER SOLUTIONS**

## V. CONCLUSIONS

We investigated by how much do location solutions change when different objectives are used and demand distribution change. When demand is uniformly distributed over a large area, we show that the solution of locating multiple facilities in the area is the same for many location objectives. However, when demand is not uniformly distributed, we show in two case studies that the location for minimizing the weighted sum of distances may be quite far from the location that minimizes the maximum distance.

The *p*-median objective is to minimize the weighted sum of distances between demand points and their closest facility. The *p*-center objective is to minimize the maximum distance between each demand point and its closest facility. The 1-median and 1-center problems were solved on a global scale assuming that demand is generated in the 40 most populated metropolitan areas in the world. The 1-median and 1-center locations are far from one another. They are more than 3,800 miles apart.

Using data about 477 municipalities in California shows that *p*-median solutions are near large metropolitan areas while p-center solutions tend to be in different regions of the state. For example, the 3-median locations are in the three large metropolitan areas, San Francisco, Los Angeles and San Diego in the western part of the state while the 3-center locations are inland in the eastern part of the state. The median and center solutions resulted in significantly different locations when populations are not uniformly distributed.

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## APPENDIX: SPHERICAL DISTANCES

Two points in latitudes $\varphi_1, \varphi_2$ and longitudes $\theta_1, \theta_2$ are located on the surface of a sphere of radius $R$. The great circle distance formula, $d_{sph}$, is (Drezner and Wesolowsky, 1978):

$$d_{sph} = R \arccos\{\cos\varphi_1 \cos\varphi_2 \cos(\theta_1 - \theta_2) + \sin\varphi_1 \sin\varphi_2\} \quad (3)$$

Another way to calculate the distance is to convert the points on the surface of the sphere to three-dimensional coordinates $(x, y, z)$ so that $x^2 + y^2 + z^2 = R^2$. A point $(\varphi, \theta)$ is transformed to:

$$x = R\cos\varphi \sin\theta;\ y = R\cos\varphi\cos\theta;\ z = R\sin\varphi. \quad (4)$$

The reverse transformation is:

$$\varphi = \arcsin\frac{z}{R};\quad \theta = \arctan\frac{x}{y} \text{ with the appropriate quadrant selected} \quad (5)$$

The distance between the points in a three dimensional space is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

The angle of the segment seen from the sphere's center is $\theta = 2\arcsin\frac{d}{2R}$. Therefore, the great circle distance is:

$$d_{sph} = R\theta = 2R\arcsin\frac{d}{2R}$$

because the great circle is on the plane connecting the two points and the center of the sphere. The center of a spherical segment is obtained by finding the center of the segment connecting the points in three dimensional space and dividing each coordinate by the distance from the sphere's center.