The JRP with Multiple Replenishment Sources and Fill Rates

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This paper extends the Joint Replenishment Problem (JRP) to a stochastic demand environment where the demand service is measured by fill rates. This paper also considers replenishment from multiple locations. The relevant costs include family order costs, item order costs, and inventory holding costs for both cycle and safety stocks. Safety stock costs are explicitly considered in the formulation, as their holding costs vary nontrivially with the model's decision variables.

An efficient solution procedure is developed for this model. Properties of the non-convex feasible space are identified and used in the solution approach. The solution to the mathematical model is comprised of the basic period length, the family multipliers, and the item multipliers that give the lowest total cost of placing orders and carrying inventory. The family multipliers and items multipliers are restricted to integer-powers-of-two.

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I. INTRODUCTION

In this paper, the joint replenishment problem (JRP) with safety stocks held for a fill rate criterion is studied. The fill rate criterion requires a percentage of demand for each item to be met. Our problem environment is further characterized by: The pr-MF-JRP is developed under the following assumptions.

- Time-stationary, normally distributed demand with known parameters
- Multiple product families, each family representing a supplier location
- Minor order cost for each product within the family
- Replenishment to stock, with safety stock required to ensure a minimum fill-rate requirement
- Relevant costs include:
  - Major order costs
  - Minor order costs
  - Cycle inventory holding costs
  - Safety stock inventory holding costs

Fill rates are popular in industry as a measure of customer service. The behavior of fill-rate safety stock levels with respect to the length of time for which it is held differs from safety stock for the service level criterion. Whereas the latter increases with respect to time, required safety stock levels for fill rates can increase, decrease or increase and then decrease with time in an interval of interest.

In §II., we examine the relevant JRP literature to date and justify our contribution in that context. We explore the behavioral differences between the fill-rate safety stock and service-level safety stock in §III. with a discussion of the computational relationship between the service level and the fill rate criteria. We introduce the The Multi-Family JRP with Fill-Rate Safety Stock in §IV.. In §V., we derive the cost function for the single item problem. We formulate the problem and discuss its solution properties. In §VI., we formulate the single Family problem; we derive a continuous relaxation, which will serve as a step in the solution procedure and as a lower bound. The multi-family algorithm for fill rates is developed in §VII.. Examples are given in §VIII. followed by an evaluation of the algorithms in §IX.. We provide concluding remarks in §X..

II. LITERATURE REVIEW

The literature review is summarized in Table 1. Goyal (1974), Andres & Emmons (1976), Joneja (1989), Federgruen & Zheng (1992), Fung & Ma (2001), Viswanathan & Ma (2002), Moon & Cha (2006), and Praharsi et al. (2010), are examples of Deterministic JRP research, with the latter most offering an optimal algorithm. Robinson & Lawrence (2004) add a production capacity constraint to their JRP model. Recent research on replenishing items with highly correlated demand include Zhang et al. (2011) and Wang et al. (2011).

A more recent stream of research, e.g. Eynan & Kropp (1997), Tagaras & Vlachos (2002), and Karalli (2011), extends the JRP to stochastic demand environments with safety stock held to hedge against stock outs. These efforts represent attempts to solve models that more closely resemble real-world conditions. A more popular service measure in industry, particularly in retail environments, is the fill rate measure of service, which was addressed by Rao (2003) for the single product case.

The single product case, though not straightforward, is much easier to solve than situations which require the coordination of multiple products and multiple families. The study of fill-rate safety stock in a multi-family, multi-family is required in order to manage inventory policies and associated costs as they actually occur in practice.

The contribution of this paper is the last entry in Table 1. This study will contribute to the JRP literature by considering the effect of fill-rate safety stock costs on the joint ordering decisions that faces the management of a firm that replenishes from multiple locations. The environment considered is that of multiple items partitioned into multiple families. The manager's challenge is to balance competing costs in order to achieve a low cost replenishment solution. The family and item fixed costs compete against the variable costs of holding working stock and safety stock inventories. These costs make up the relevant costs of the problem, the sum of which will be referred to as the total cost. Even though this problem is intended to address multiple families, it is also applicable to the single family and the single item problems.

Examples of recent efforts in studying the JRP with multiple suppliers include Moon et al. (2008) and Cha & Park (2009), whose focus is primarily on incorporating quantity discounts into their deterministic models.

III. Fill-Rate Safety Stock

3.1 The Fill Rate

Consider an item, whose demand is normally distributed with mean d units per period and standard deviation of demand σ units per period, is replenished every t periods, with a replenishment lead-time $L \geq 0$.

When considering the fill rate, one seeks to determine a product's expected excess demand, which cannot be filled from inventory so that (1) below holds, where the LHS, f, is the required fill rate for the item under consideration.

$$f = \frac{q}{q+e} \tag{1}$$

The quantity on hand, q, at the beginning of the inventory cycle is defined in (2). The first term of the right hand side of (2) is the mean demand between replenishments. The second term of the right hand side of (2) is the safety stock required to meet the specified fill rate, f.

$$q = dt + z\sigma\sqrt{L+t} \tag{2}$$

The expected shortage, e, given $z\sigma\sqrt{L+t}$ units of safety stock, is defined in (3).

$$e = E(z)\sigma\sqrt{L+t} \tag{3}$$

where, $E(z)$, given in (4), is the partial expectation evaluated at z.

$$E(z) = \frac{1}{2\pi}\int_z^\infty (x-z)\,e^{-\frac{x^2}{2}}dx \tag{4}$$

To compute e, first find z in (4) below so that (1) holds, using the identities in (2) and (3). Since e is the demand expected to exceed q, the denominator in (1) is the expected demand given q units were in stock.

The partial expectation in (4) can be expressed as follows Brown (1967):

$$E(z) = \phi(z) - zF(z) \tag{5}$$

where,

$\phi(z)$	Standard normal p.d.f. evaluated at z
$\Phi(z)$	Area under the standard normal curve to the left of z
$F(z)$	Area under the standard normal curve to the right of z; that is $F(z) = 1 - \Phi(z)$

We combine (1), (3), (2), and (5) to yield (6), below.

$$[\phi(z) - zF(z)] - \left(\frac{1-f}{f}\right)\left(\frac{dt}{\sigma\sqrt{L+t}} + z\right) = 0 \tag{6}$$

We can now express t in terms of z as follows in (7) below

$$\frac{t}{\sqrt{L+t}} = \left[\left(\frac{f}{1-f}\right)E(z) - z\right]\left(\frac{\sigma}{d}\right) \tag{7}$$

Property 1. An increase (decrease) in t requires a decrease (increase) in z to maitain the identitly in (1).

Proof. Using (1) and (3), we can write (8) below.

$$\frac{dt + z\sigma\sqrt{L+t}}{dt + z\sigma\sqrt{L+t} + E(z)\sigma\sqrt{L+t}} = f \tag{8}$$

Because $\frac{\partial E(z)}{\partial z} < 0$ Brown (1967), maintaining the identity in (8) requires a reduction in the value of z to raise the value of the denominator of the LHS of (8) when t, in the numerator, increases. □

The relationship of z to t is illustrated in Fig. 1. The demand line is the demand rate multiplied by the cycle time, dt. When $z < 0$, the order quantity, q, required to satisfy f is smaller than the mean demand over the time until the next replenishment. To satisfy the identity in (6), the value of z is zero when

$$\frac{t}{\sqrt{L+t}} = \frac{1}{\sqrt{2\pi}}\left(\frac{\sigma}{d}\right)\left(\frac{f}{1-f}\right). \tag{9}$$

This can be seen in Fig. 1, where at $z = 0$, the total stock curve, q, intersects with the demand line, dt, and then falls below dt, when $z < 0$.

3.2 Service Level vs. Fill Rate Safety Stock Inventories

Consider a product with a normally distributed daily demand with a mean of 1000 units and a standard deviation of 400 units. If safety stock were held for a period up to, say, 30 days to hedge against stock outs, the level would vary with time according to the expression $400z\sqrt{t}$ where z is the standard normal variate corresponding to one minus the probability that a stock out will occur in the time interval $t \in (0, 30)$. The level of service level safety stock required to cover a specific length of time in which we wish to hedge against a stock out due to unexpected demand is displayed in Fig. 2. Each curve in Fig. 2 represents a different service level (one minus the probability of incurring a stock out), specifically 95%, 97%, 99%, and 99.99%.

The graph for safety stock levels for 95%, 97%,

Table 1: Contributions of the Literature and this Paper

Author(s)	Year	Replenishment Type	Safety Stock
Praharsi et al.	(2010)	MF	
Wang et al.	(2011)	MF	QD
Cha & Park	(2009)	MF	QD
Moon et al.	(2008)	SF	QD
Moon & Cha	(2006)	SF	R
Robinson & Lawrence	(2004)	SF	
Rao	(2003)	SI	FR
Tagaras & Vlachos	(2002)	SF	SL
Viswanathan & Ma	(2002)	SF	
Karalli	(2011)	MF	SL
Fung & Ma	(2001)	SF	
Eynan & Kropp	(1997)	SF	SL
Federgruen & Zheng	(1992)	SF	
Joneja	(1989)	SF	
This Paper		MF, SF, SI	FR

Legend:	
SI	single-item
SF	single-family, multi-item
MF	multi-family, multi-item
FR	fill rate
SL	service level
R	resource restrictions
QD	quantity discounts

99% and 99.99% fill rate requirements is in Fig. 3 and presents a different picture than Fig. 2. The curves for the 95% and 97% fill rates first increase, then decrease. The reason for the change in direction is that, unlike service level safety stock, where the standard normal variate is static for each curve, the standard normal variate in fill rates is itself, a decreasing function of t. By inspection of (6) above, a change in the period length, t, over which we wish to hedge against unexpected demand, would necessitate a change in z to maintain the equation's identity.

The value of z can become negative, as is the case for the 95% and 97% fill-rate curves.

3.3 Service Level vs. Fill Rate Safety Factors

The service level safety factor (or the control variate) is constant over all values of t. That is not the case for the fill-rate.

Property 2. (The Fill-Rate Safety Factor) $\forall f \in (0,1), \exists! z^f(0) \ni \left(\frac{f}{1-f}\right) E(z^f(0)) - z^f(0) = 0$ (see (7)).

Proof. That is, when $z = z^f(0)$, the value of t is 0. Values of $z \geq z^f(0)$ imply $\sqrt{t} \leq 0$ and are therefore not in the domain of values of z for f. □

Property 2 is illustrated in Fig. 4. Property 2 will be useful in the development of an efficient solution algorithm.

IV. THE MULTI-FAMILY JRP WITH FILL-RATE SAFETY STOCK

We now present a framework that serves as a generalization of (a) multi-family JRP with fill-rate safety stock (MFJRP-FR), (b) the (single-family) JRP with Fill-Rate Safety Stock (JRP-FR), and (c) the (single-item) EOQ with Fill-Rate Safety Stock.

Problem Environment. The MFJRP-FR is a continuous-time, infinite horizon extension of the JRP in which a retail firm replenishes its inventory from multiple suppliers. Each supplier represents a family. The retailer needs to coordinate shipment arrivals from all the suppliers with its own resources, such as personnel and dock space.

There are n families, each identified by a subscript $i \in \mathbb{N} = 1, \ldots, n$. Each item is identitified by a unique family-item pair,$(i, j) \in \mathbb{N} \times \mathbb{N}$, with the subscript $j \in \mathbb{N}$, denothing the j^{th} item in family i. The use of n as both the number of families and the number of items within each family is a notational convenience. When the number n exceeds the actual number of items in any family, or exceeds the number of families, dummy items and/or families are simply created and the value of zero is assigned to their parameters.

- For each family, there is:
  - a known sequence-independent order cost
  - a known delivery lead-time
- For each item, the demand is:
  - time-stationary
  - normally distributed, with
  - known mean and standard deviation
  - uncorrelated with other items
  - not substitutable with other items
- For each item, there is:
  - a known sequence-independent ordering cost
  - a known cost of storing the item in inventory
  - a specified fill rate

Safety stock is maintained for each item in order to meet the specified fill rates. We further assume items do not have multiple suppliers; that is, every item is unique. The relevant costs include a fixed cost for ordering the item and a holding cost per unit per unit of time for both cycle and safety stocks. The objective is to minimize the total average relevant cost per unit time in a cyclic schedule.

4.1 Notation

Basic Notation

$\mathbb{W}$ $= \{0, 1, 2, \ldots\}$; the set of non-negative integers
$\mathbb{P}$ $= \{2^p : p \in \mathbb{W}\}$; the set of integer-powers-of-two

Problem Parameters

A_i order cost incurred when one or more items are ordered from $i \in \mathbb{N}$
L_i Delivery lead-time for family $i \in \mathbb{N}$
a_{ij} order cost incurred when item (i, j) is ordered
d_{ij} demand mean for item (i, j)
σ_{ij} demand standard deviation for item (i, j)
h_{ij} inventory carry cost per unit per unit of time for item (i, j)
f_{ij} required fill rate for item (i, j)

Decision Variables

z_{ij} safety factor for item (i, j)
t_{ij} length of the basic period (BP) for item (i, j)
T length of the basic period (SF and MF problems)
K_i multiplier for family i; $K_i \in \mathbb{P}$
k_{ij} multiplier for item (i, j) ; $k_{ij} \in \mathbb{P}$

Additional Notation

$\mathbf{K}$ $n-$vector
$\mathbf{k}$ $n \times n-$matrix
$\mathbf{k}_{i\cdot}$ n^{th} row of $\mathbf{k}$
$\mathbf{z}$ $n \times n-$matrix
$c(\cdot)$ cost function evaluated at $(\cdot)$
b_{ij} $= \frac{1}{2} h_{ij} d_{ij}$

$$g_{ij} = \begin{cases} h_{ij} z_{ij} \sigma_{ij} & \text{if } z_{ij} \geq 0 \\ \frac{1}{2} h_{ij} z_{ij} \sigma_{ij} & \text{otherwise} \end{cases}$$

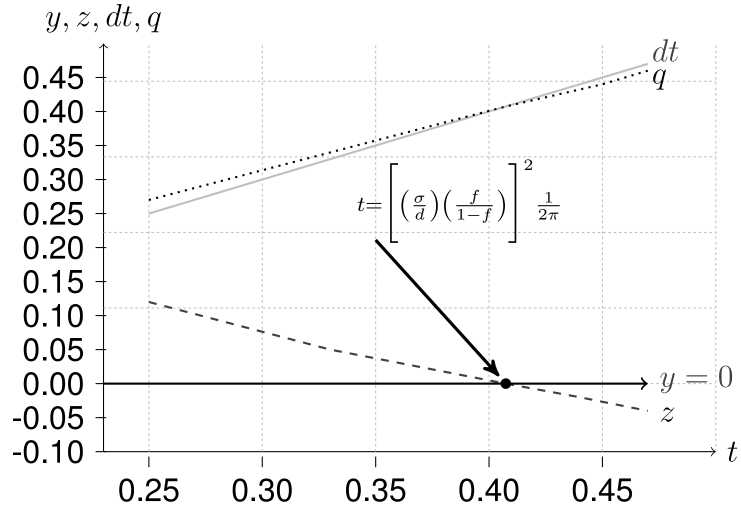

Figure 1: The relationship of z to t that satisfies the identity in (6), when d, σ, and f are held fixed. In this example, $L = 0$, $d = 1$, $\sigma = 0.4$, and $f = 0.80$; when $t = 0.4074$, $z = 0$. The horizontal line crossing the origin is added for emphasis.

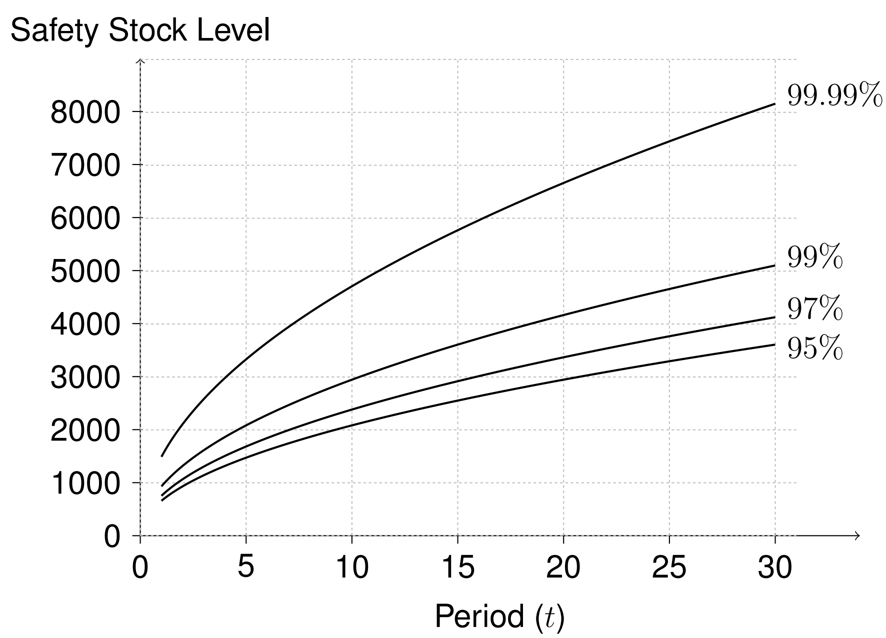

Figure 2: Requirements of service level safety stock as a function of time for an item with $d = 1000$ units per period, and $\sigma = 400$ units per period. Because z is constant for any given service level, the safety stock function is always increasing in t.

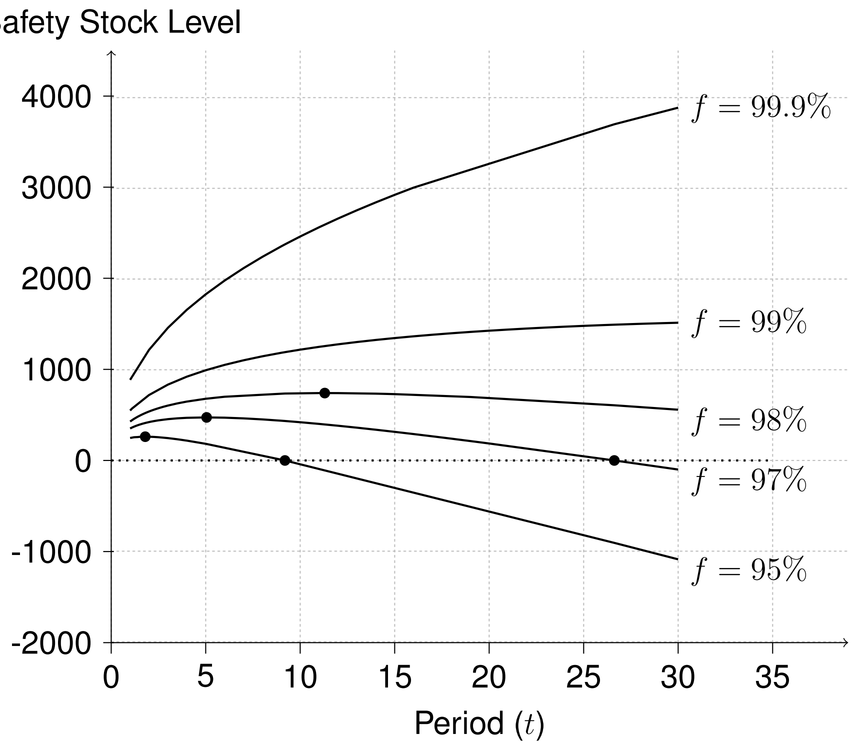

Figure 3: Requirements of fill-rate safety stock as a function of time for an item with $d = 1000$ units per period, and $\sigma = 400$ units per period. Because z is a decreasing function of t, the safety stock function increases with t until $t = \left[\left(\frac{\sigma}{d}\right)\left(\frac{f}{1-f}\right)\right]^2 \frac{1}{2\pi}$ and decreases thereafter.

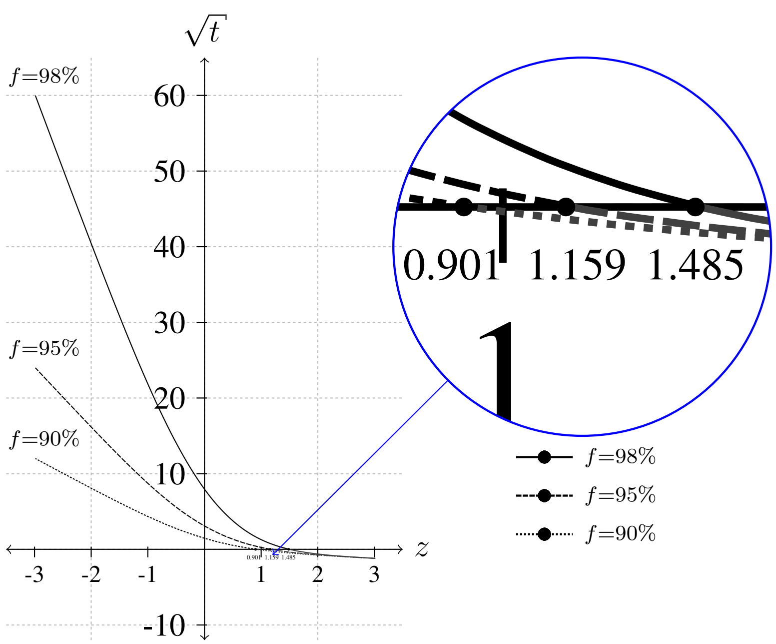

Figure 4: Upperbound of the safety factor, z, determined exclusively by the fill rate, f. Safety factor curves for $f = 90\%$ (dotted), 95% (dashed), and 98% (solid) displayed for an item with $d = 1000$ and $\sigma = 400$. When $t = 0$, $z_0^{90\%} = 0.901$, $z_0^{95\%} = 1.159$, and $z_0^{98\%} = 1.485$.

$$\begin{aligned} T_i &= TK_i \\ T_{ij} &= TK_i k_{ij} \\ x_i &= TK_i \\ y_{ij} &= TK_i k_{ij} \end{aligned}$$

4.2 Derivation of the Cost function

The problem parameters are known for each family of items and its items. The firm would like to adopt an inventory policy that (*a*) minimizes the total relevant operating costs (*b*) is feasible to implement, and (*c*) complies with its service criteria. The average family order cost per unit time of the item is given in (10). The average item order cost per unit time of the item is given in (11).

$$\sum_{i\in\mathbb{N}} \frac{A_i}{T_i} \tag{10}$$

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \frac{a_{ij}}{T_{ij}} \tag{11}$$

The average working stock holding cost per unit time when the BL length is T units and item (i, j) is replenished every $TK_i k_{ij}$ time units, is given in (12).

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} b_{ij}T_{ij} \tag{12}$$

The total average safety stock holding cost per unit time is given in (13).

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} g_{ij}\sqrt{L_i + T_{ij}} \tag{13}$$

When $z_{ij} \geq 0$ the value of g_{ij} in (13) is positive (see **Additional Notation**). It reflects the fact that, on average, this is the quantity that is expected to remain at the end of the cycle t. When $z_{ij} < 0$, $g_{ij} < 0$, the working stock quantity, as defined in (12), is less than the expected demand over the cycle T_{ij}. The expectation, then, is for the inventory to be exhausted at the start of the following cycle and the cost of carry is reduced.

The total average holding cost per unit time is the sum of (11) and (13).

4.3 Problem Formulation

The firm's objective is to find $(T, \mathbf{K}, \mathbf{k}, \mathbf{z})$ so as to minimize the total average cost, given in (14). To ensure that the safety stock quantities are set to levels that satisfy the required fill rate for each item, we require that the identity in (15) be enforced by making it a constraint as shown in (18). The constraints in (16) and (17) require that the family and item multipliers be integer-powers-of-two (IPOT). While a strictly positive value of T is sought, the strict inequality can be written as a weak one as shown in (18) without loss of accuracy.

$$c\left(T, \mathbf{K}, \mathbf{k}, \mathbf{z}\right) = \sum_{i\in\mathbb{N}} \frac{A_i}{T_i} + \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \left(\frac{a_{ij}}{T_{ij}} + b_{ij}T_{ij} + g_{ij}\sqrt{L_i + T_{ij}}\right) \tag{14}$$

$$f\left(f_{ij}, d_{ij}, \sigma_{ij}, T_{ij}\right) = \frac{T_{ij}}{\sqrt{L_i + T_{ij}}} - \left(\frac{\sigma_{ij}}{d_{ij}}\right)\left[E(z_{ij})\left(\frac{f_{ij}}{1 - f_{ij}}\right) - z_{ij}\right] = 0 \tag{15}$$

Problem M. The objective of Problem M is to find $(T, \mathbf{K}, \mathbf{k}, \mathbf{z})$ so as to

$$\begin{aligned} &\text{Minimize} && (14) && \\ &\text{Subject to} && (15); && \forall\,(i,j) \in \mathbb{N}\times\mathbb{N} \\ &&& K_i \in \mathbb{P}; && \forall\, j \in \mathbb{N} \quad (16) \\ &&& K_{ij} \in \mathbb{P}; && \forall\,(i,j) \in \mathbb{N}\times\mathbb{N} \quad (17) \\ &&& T \geq 0 && \quad (18) \end{aligned}$$

Karalli & Flowers (2006) show that the solution to a problem, whose structure is similar to ours, can always be represented in anchor form (AF). The AF property facilitates the search for a solution to problem F. A solution, $(T^\star, \mathbf{K}^\star, \mathbf{k}^\star, \mathbf{z}^\star)$, to the Problem M is in AF if (*a*) $0 < t \in \mathbb{R}$; (*b*) $\forall i \in \mathbb{N}, K_i \in \mathbb{P}$; (*c*) $\mathbf{K}^\star = (K_1, K_2, \ldots, K_n)$, with $K_1 = 1 \leq K_1 \leq \cdots \leq K_n$; (*d*) $\forall\,(i, j) \in \mathbb{N}\times\mathbb{N}$, $k_{ij} \in \mathbb{P}\times\mathbb{P}$; and (*e*) $\mathbf{k}_{i\cdot} = (k_{i1}, k_{i2}, \ldots, k_{in})$, with $k_{i1} = 1 \leq k_{i2} \leq \cdots \leq k_{in}$.

4.4 Lower Bound for Problem M

The following continuous relaxation, problem M_R, serves as a lower bound of problem M. In order to relax the IPOT restrictions on the K_is and k_{ij}s, we define $x_i \equiv TK_i$ and $y_{ij} \equiv TK_ik_{ij}$. The objective function for problem M_R becomes

$$c(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{i \in \mathbb{N}} \frac{A_i}{x_i} + \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \left(\frac{a_{ij}}{y_{ij}} + b_{ij} y_{ij} + g_{ij} \sqrt{L_i + y_{ij}} \right) \quad (19)$$

Constraint (15) becomes

$$f(f_{ij}, d_{ij}, \sigma_{ij}, y_{ij}) = \frac{y_{ij}}{\sqrt{L_i + y_{ij}}} - \left(\frac{\sigma_{ij}}{d_{ij}} \right) \left[E(z_{ij}) \left(\frac{f_{ij}}{1 - f_{ij}} \right) - z_{ij} \right] = 0 \quad (20)$$

Problem M_R The objective of Problem M_R is to find $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ so as to

$$\begin{aligned} &\text{Minimize} && (19) && \\ &\text{Subject to} && (20); && \forall i, j \in \mathbb{N} \\ & && x_i - y_{ij} \le 0; && \forall i, j \in \mathbb{N} \quad (21) \\ & && x_i, y_{ij} \ge 0; && \forall i, j \in \mathbb{N} \quad (22) \end{aligned}$$

V. The Single Item Problem

Solving the single item problem is the departure point for solving the single-item, single-family, and multi-family problems.

Problem S Find the cycle time, y_{ij}^*, and the safety factor, z_{ij}^*, for the item so as to

$$\begin{aligned} &\text{Minimize} && c(y_{ij}, z_{ij}) = \frac{a_{ij}}{y_{ij}} + b_{ij} y_{ij} + g_{ij} \sqrt{L_i + y_{ij}} && \quad (23) \\ &\text{Subject to} && (20); \quad \forall i, j \in \mathbb{N} && \\ & && y_{ij} \ge 0; \quad \forall i, j \in \mathbb{N} && \quad (24) \end{aligned}$$

Solution to Problem S The steps for solving Problem S are

1. find z_0^f using the Newton-Raphson Method given in the pr-MF-JRP Algorithm, Step 1
2. perform the golden section line search along $z \in \left[a = -3, b = z_0^f \right]$ given in the pr-MF-JRP Algorithm, Step 2

For a tolerance of $\epsilon = 0.0001$, the pr-MF-JRP Algorithm, Step 1 usually requires five to seven iterations to provide an answer. For a tolerance of $l = 0.0001$, the pr-MF-JRP Algorithm, Step 2 will solve problem S in n iterations, where n is the smallest integer satisfying the inequality in (25) Bazaraa et al. (2006).

$$(0.618)^{n-1} \le \frac{l}{b - a} \quad (25)$$

For fill-rates of 92% and higher, $n = 24$. Below 92%, $n = 23$.

VI. The Single Family Problem

Problem M_R can be separated into n single family problems, as shown in Problem F_R.

Problem F_R The objective of Problem F_R is to find $(x_i, \mathbf{y}_{i\cdot}, \mathbf{z}_{i\cdot})$ for some $i \in \mathbb{N}$ so as to

$$\text{Minimize } c(x_i, y_{ij}, z_{ij}) = \frac{A_i}{x_i} + \sum_{j \in \mathbb{N}} \left(\frac{a_{ij}}{y_{ij}} + b_{ij} y_{ij} + g_{ij} \sqrt{L_i + y_{ij}} \right) \quad (26)$$

$$\begin{aligned} &\text{Subject to} && (20); && \forall i, j \in \mathbb{N} \\ & && x_i - y_{ij} \le 0; && \forall i, j \in \mathbb{N} \quad (27) \\ & && x_i, y_{ij} \ge 0; && \forall i, j \in \mathbb{N} \quad (28) \end{aligned}$$

The existence of a solution is proven using the same approach given by Karalli & Flowers (2006).

Proposition 3. *Given an ordered set of item subscripts $\mathbb{N}$, so that for $\mathbb{N} \ni j = 1, 2, \ldots, n$, $y_1 \le$*

$y_2 \leq \cdots \leq y_n$, *if* $(x_i^\star, \mathbf{y}_{i\cdot}^\star, \mathbf{z}_{i\cdot}^\star)$ *is a Karush-Kuhn-Tucker (KKT) point for an instance of problem* F_R*, then* $x_i = y_{i1} = \cdots = y_{im}$ *for some* $m \in \mathbb{N}$*, with* $m \leq n$*.*

Proof. We proceed with a proof by contradiction. If $(x_i^\star, \mathbf{y}_{i\cdot}^\star, \mathbf{z}_{i\cdot}^\star)$ were a KKT point and $\forall m \in \mathbb{N}, t < y_m$ then $u_j = 0, \forall j \in \mathbb{N}$ which would preclude the gradient condition shown in (29) from having a real solution.

$$-\frac{A_i}{x_i^2} + \sum_{j \in \mathbf{N}} u_{ij} = 0, \tag{29}$$

Moreover, since m has n possible values, there are at most n KKT points. □

Problem F. Problem F is the formulation for the single family problem. The objective of Problem F is to find the cycle time, T, the multipliers k_j , and the safety factor z_j for each item $j \in \mathbb{N}$ so as to

$$\text{Minimize} \quad C(T, z, \mathbf{k}) = \frac{A}{T} + \sum_{j \in \mathbb{N}} \left(\frac{a_j}{Tk_j} + b_j T k_j + g_j \sqrt{L + Tk_j} \right) \tag{30}$$

$$\text{Subject to} \quad T \geq 0 \tag{31}$$

$$K_j \in \mathbb{P}; \quad \forall j \in \mathbb{N} \tag{32}$$

$$(20); \quad \forall j \in \mathbb{N} \tag{33}$$

Algorithm 1 Procedure to find z_t^f using the Newton-Raphson Method

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Require:

f, d, σ //item preperties

t //to find z_0^f set $t = 0$

1: **Initialize**

$\epsilon \leftarrow 0.0001$

$z_0 \leftarrow 3$

2: **Function and Its Derivative**

$f(z) = \left[\left(\frac{f}{1-f}\right) E(z) - z\right] \left(\frac{\sigma}{d}\right) - \frac{t}{\sqrt{L+t}}$

$f'(z) = \left[-\left(\frac{f}{1-f}\right) F(z) - 1\right] \left(\frac{\sigma}{d}\right)$

Begin Newton-Raphson Method

3: $z_1 = z_0 - \frac{f(z_0)}{f'(z_0)}$

4: **while** $|z_1 - z_0| > \epsilon$ **do**

5: $\quad z_0 = z_1$

6: $\quad z_1 = z_0 - \frac{f(z_0)}{f'(z_0)}$

7: **end while**

8: $z^f(t) = z_1$

9: **return** $z^f(t)$

End

Algorithm 2 Solution Procedure for the Single Item Problem

1: **Initialize**

$$\beta \leftarrow \frac{\sqrt{5}-1}{2} \qquad \alpha \leftarrow 1-\beta$$

$$\varepsilon \leftarrow 0.0001 \qquad (z_a^0, z_b^0) \leftarrow \left(-3, z^f(0)\right)$$

2: **Return Values**

$$z^\star \leftarrow \frac{(z_a^k + z_b^k)}{2}$$

$$t^\star \leftarrow \frac{\gamma^2}{2} + \frac{1}{2}\sqrt{\gamma^4 + 4\gamma^2 L} \qquad \text{where} \qquad \gamma \leftarrow \left(\frac{\sigma}{d}\right)\left[E(z^\star)\left(\frac{f}{1-f}\right) - z^\star\right]$$

$$c(t^\star, z^\star) \leftarrow \frac{s}{t^\star} + bt^\star + g^\star\sqrt{L+t^\star} \qquad \text{where} \qquad g^\star \leftarrow \begin{cases} hz^\star\sigma & \text{if } z^\star \geq 0 \\ \frac{1}{2}hz^\star\sigma & \text{otherwise} \end{cases}$$

3: $(z_a^1, z_b^1) = (z_a^0, z_b^0)$

4: **for** $k = 1$ to 24 **do**

5: $\quad \delta^k \leftarrow z_b^k - z_a^k$

$$\left(\lambda^k, \mu^k\right) \leftarrow \left(z_b^k + \alpha\delta^k, z_b^k + \beta\delta^k\right)$$

$$\delta^k \leftarrow \mu^k - \lambda^k$$

6: $\quad$ **if** $\delta^k \leq \varepsilon$ **then**

7: $\qquad$ **return** $z^\star$, $t^\star$, and $c(t^\star, z^\star)$ Using the equations in statement 2

$\qquad$ //**Exit** algorithm

8: $\quad$ **else**

9:

$$t_\lambda^k \leftarrow \frac{(\gamma_\lambda^k)^2}{2} + \frac{1}{2}\sqrt{(\gamma_\lambda^k)^4 + 4(\gamma_\lambda^k)^2 L} \qquad t_\mu^k \leftarrow \frac{(\gamma_\lambda^k)^2}{2} + \frac{1}{2}\sqrt{(\gamma_\lambda^k)^4 + 4(\gamma_\lambda^k)^2 L}$$

$$\text{where } \gamma_\lambda^k \leftarrow \left(\frac{\sigma}{d}\right)\left[E(\lambda^k)\left(\frac{f}{1-f}\right) - \lambda^k\right] \qquad \text{where } \gamma_\mu^k \leftarrow \left(\frac{\sigma}{d}\right)\left[E(\mu^k)\left(\frac{f}{1-f}\right) - \mu^k\right]$$

$$c\left(\lambda^k\right) \leftarrow \frac{s}{t_\lambda^k} + \frac{1}{2}hdt_\lambda^k + g_\lambda^k\sqrt{t_\lambda^k} \qquad c\left(\mu^k\right) \leftarrow \frac{s}{t_\mu^k} + \frac{1}{2}hdt_\mu^k + g_\mu^k\sqrt{t_\mu^k}$$

$$\text{where } g_\lambda^k \leftarrow \begin{cases} h\lambda^k\sigma & \text{if } \lambda^k \geq 0 \\ \frac{1}{2}h\lambda^k\sigma & \text{otherwise} \end{cases} \qquad \text{where } g_\mu^k \leftarrow \begin{cases} h\mu^k\sigma & \text{if } \mu^k \geq 0 \\ \frac{1}{2}h\mu^k\sigma & \text{otherwise} \end{cases}$$

10:

$$\left(z_a^k, z_b^k\right) \leftarrow \begin{cases} \left(\lambda^k, z_b^k\right) & \text{if } c\left(\lambda^k\right) > c\left(\mu^k\right) \\ \left(z_a^k, \mu^k\right) & \text{otherwise} \end{cases}$$

11: $\quad$ **end if**

12: **end for**

13: **return** $z^\star$, $t^\star$, and $c(t^\star, z^\star)$ Using the equations in statement 2

//**Exit** algorithm

Solution Steps for the Single Family Problem

The steps for solving Problem F are

1 solve Problem S n times; for each item $j = 1, \ldots, n$ run

- the pr-MF-JRP Algorithm, Step 1, and
- the pr-MF-JRP Algorithm, Step 2

2 employ the pr-MF-JRP Algorithm, Step 3 to

- find the KKT points of the continuous relaxation of Problem F
  - The number of KKT points is between 1 and n
- select the KKT point, $m^\star$, with the lowest objective value, $c_{m^\star}(t^\star, \mathbf{y}^\star)$.
  - $c_{m^\star}(t^\star, \mathbf{y}^\star)$ is a lower bound for Problem F.

3 employ the pr-MF-JRP Algorithm, Step 5 to execute the roundoff algorithm to

- compute n trial values of $(T, \mathbf{k})$
- select the cost minimizing $(T^\star, \mathbf{k}^\star)$

VII. The Multiple Family Algorithm

The steps for solving Problem F are

1 solve Problem S n^2 times; for each item $(i, j) \in \mathbb{N} \times \mathbb{N}$ run

- the pr-MF-JRP Algorithm, Step 1, and
- the pr-MF-JRP Algorithm, Step 2

2 employ the pr-MF-JRP Algorithm, Step 3 n^2 times; for each family $i \in \mathbb{N}$

- find the KKT points of the continuous relaxation of Problem F
- select the KKT point, $m^\star$, with the lowest objective value, $c_i^{m^\star}(x_i^\star, \mathbf{y}^\star)$.
  - $\sum_{i \in \mathbb{N}} c_i^{m^\star}(x_i^\star, \mathbf{y}^\star)$ is a lower bound for Problem F.

3 employing the pr-MF-JRP Algorithm, Step 6, execute the multi-family roundoff algorithm to

- compute n^2 trial values of $(T, \mathbf{K}, \mathbf{k})$
- select the cost minimizing $(T^\star, \mathbf{K}^\star, \mathbf{k}^\star)$

VIII. Examples

8.1 Single Item Example

Data

$d = 294$ $\quad \sigma = 143.1100$
$h = \$0.18$ $\quad f = 0.9038$
$a = \$143.43$ $\quad L = 5.1706$

Solution

$z^\star = 0.0561$ $\quad t^\star = 2.7859$
$c(t^\star, z^\star) = \$128.90$

Upper Bound

$z_{ub} = 0.1101$ $\quad t_{ub} = 2.3338$
$c(t_{ub}, z_{ub}) = \$130.63$
savings 1.3487%

8.2 Single Family Example

Data

$A = \$486.09$ $\quad L = 1.7616$

$d = \begin{bmatrix} 284 & 256 & 228 & 244 & 494 \end{bmatrix}$

$\sigma = \begin{bmatrix} 140.13 & 88.93 & 70.49 & 106.79 & 201.37 \end{bmatrix}$

$h = \begin{bmatrix} \$0.12 & 0.17 & 0.14 & 0.17 & 0.26 \end{bmatrix}$

$f = \begin{bmatrix} 0.9500 & 0.9364 & 0.9407 & 0.9026 & 0.9787 \end{bmatrix}$

$a = \begin{bmatrix} \$130.28 & 140.84 & 123.47 & 132.43 & 77.61 \end{bmatrix}$

Solution

$z^\star = 0.5824 \qquad t^\star = 2.0119$
$c\left(t^\star, k^\star, z^\star\right) = \$956.61$
$k = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \end{bmatrix}$

Lower bound

$z_{lb} = 0.5824 \qquad t_{lb} = 2.8116$
$c\left(t_{lb}, k_{lb}, z_{lb}\right) = \$905.75$
Performance: 5.62% above the lower bound.

8.3 Multi-Family Example

Data

$$S = \begin{bmatrix} \$276.63 & 407.97 & 220.08 & 365.42 & 466.37 \end{bmatrix}$$

$$L = \begin{bmatrix} 1.3848 & 1.9996 & 2.4608 & 1.3720 & 0.3250 \end{bmatrix}$$

$$d = \begin{bmatrix} 148 & 201 & 357 & 432 & 428 \\ 257 & 78 & 74 & 200 & 199 \\ 475 & 197 & 482 & 67 & 392 \\ 320 & 398 & 176 & 391 & 388 \\ 366 & 88 & 337 & 389 & 156 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} 60.47 & 63.41 & 170.59 & 208.49 & 130.15 \\ 118.19 & 38.72 & 28.17 & 87.05 & 58.67 \\ 221.61 & 93.79 & 130.52 & 32.23 & 114.35 \\ 124.03 & 115.41 & 84.23 & 138.32 & 125.50 \\ 93.05 & 25.48 & 166.50 & 120.98 & 40.82 \end{bmatrix}$$

$$h = \begin{bmatrix} \$0.13 & 0.17 & 0.12 & 0.13 & 0.19 \\ 0.12 & 0.10 & 0.09 & 0.19 & 0.17 \\ 0.16 & 0.14 & 0.17 & 0.11 & 0.19 \\ 0.10 & 0.11 & 0.12 & 0.13 & 0.13 \\ 0.15 & 0.10 & 0.14 & 0.16 & 0.15 \end{bmatrix}$$

$$f = \begin{bmatrix} 0.9505 & 0.9706 & 0.9205 & 0.9101 & 0.9527 \\ 0.9876 & 0.9662 & 0.9405 & 0.9107 & 0.9393 \\ 0.9515 & 0.9889 & 0.9683 & 0.9525 & 0.9318 \\ 0.9972 & 0.9888 & 0.9010 & 0.9630 & 0.9958 \\ 0.9805 & 0.9279 & 0.9148 & 0.9246 & 0.9659 \end{bmatrix}$$

$$s = \begin{bmatrix} \$114.49 & 99.659 & 120.05 & 105.39 & 91.691 \\ 96.841 & 92.235 & 143.21 & 106.64 & 122.36 \\ 95.596 & 97.598 & 77.541 & 113.06 & 113.03 \\ 95.18 & 104.46 & 87.483 & 94.856 & 133.46 \\ 115.89 & 142.08 & 111.45 & 117.07 & 99.08 \end{bmatrix}$$

Solution

$t^\star = 2.9408$

$c\left(t^\star, K^\star, k^\star, z^\star\right) = \$3,384.38$

$$K^\star = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$k^\star = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

$$z^\star = \begin{bmatrix} 0.3501 & 0.1958 & 0.6496 & 0.2687 & 0.4770 \\ 1.2389 & 0.1792 & 0.2080 & 0.6005 & 0.09260 \\ 0.5880 & 0.5989 & 0.1503 & 1.3114 & 0.3988 \\ 0.5748 & 1.5647 & 1.1052 & 1.7691 & 0.1263 \\ 0.7062 & -0.0473 & 0.1641 & 0.3915 & -0.3614 \end{bmatrix}$$

Lower bound

$c\left(t_{lb}, z_{lb}\right) = \$3,341.70$
Performance: 1.28% above the lower bound.

Algorithm 3 Find and Test KKT Points

Initialize
  $\forall j \in \mathbb{N},\ y_j \leftarrow t_j$; re-index $\mathbf{y}$ so that $y_1 \leq y_2 \leq y_n$
  set $\mathbb{M} = \{m :$ the solution produced in iteration m is feasible$\} = \emptyset$
for $m = 1$ to n **do**
  $\bar{a}_m = A + a_1 + \cdots + a_m$
  $\bar{b}_m = b_1 + \cdots + b_m$
end for
Compute $(\bar{t}_1, \bar{z}_1)$ using the pr-MF-JRP Algorithm, Step 1 and the pr-MF-JRP Algorithm, Step 2
if $\bar{t}_1 \geq y_2$ **then**
  Constraint (33) is not satisfied; the solution is infeasible (not a KKT point).
else
  $t \leftarrow \bar{y}_1 \leftarrow \bar{t}_1$
  $\mathbb{M} \leftarrow \{1\}$
  $\mathbb{C} \leftarrow c_1(t, \mathbf{y}) = \dfrac{\bar{a}_m}{t} + \bar{b}_m t + \sum_{j=m+1}^{n} \left(\dfrac{a_j}{y_j} + b_j y_j \right) + \sum_{j \in \mathbb{N}} (g_j \sqrt{y_j})$
end if
for $m = 2$ to n **do**
  Create dummy item m_0 by duplicating item m and making one change:
    $a_m^0 \leftarrow \bar{a}_m$
  Compute (t_m^0, z_m^0) using the pr-MF-JRP Algorithm, Step 1 and the pr-MF-JRP Algorithm, Step 2
  Compute $(\bar{t}_m, \bar{z}_1, \ldots, \bar{z}_m)$ using the pr-MF-JRP Algorithm, Step 2 and the pr-MF-JRP Algorithm, Step 4
  if $\bar{t}_m < y_{m+1}$ **then**
    for $j = 1$ to m **do**
      $t \leftarrow \bar{y}_j \leftarrow \bar{t}_j$
    end for
    $\mathbb{M} \leftarrow \{m\}$
    $\mathbb{C} \leftarrow c_m(t, \mathbf{y}) = \dfrac{\bar{a}_m}{t} + \bar{b}_m t + \sum_{j=m+1}^{n} \left(\dfrac{a_j}{y_j} + b_j y_j \right) + \sum_{j \in \mathbb{N}} (g_j \sqrt{y_j})$
    //When $\bar{t}_m >= y_{m+1}$, constraint (33) is not satisfied; the solution is infeasible (not a KKT point).
  end if
end for
return $m^\star = \underset{m \in \mathbb{M}}{\operatorname{argmin}}\, \mathbb{C} = \{c_m(t, \mathbf{y}) : (t, \mathbf{y})$ is a KKT point$\}$
  //Stop–exit the pr-MF-JRP Algorithm, Step 3 and proceed to the pr-MF-JRP Algorithm, Step 5.

Algorithm 4 Golden Section Search along t

Initialize

$(t_a^0, t_b^0) \leftarrow (t_m^0, y_1)$ $\quad \delta^0 = t_b^0 - t_a^0 \quad \varepsilon \leftarrow 0.0001$

$\omega = 1 \quad \beta \leftarrow \frac{\sqrt{5}-1}{2} \quad \alpha \leftarrow 1 - \beta$

Compute ω (required number of iterations)

while $0.618^{(\omega-1)} > \frac{\varepsilon}{\delta^0}$ **do**

  $\omega \leftarrow \omega + 1$

end while

$(t_a^1, t_b^1) \leftarrow (t_a^0, t_b^0)$

for $k = 1$ to ω **do**

  $\delta^k \leftarrow t_b^k - t_a^k$

  $\left(\lambda^k, \mu^k\right) \leftarrow \left(t_a^k + \delta^k \alpha, t_b^k + \delta^k \beta\right)$

  $\delta^k \leftarrow \mu^k - \lambda^k$

  if $\delta^k \leq \varepsilon$ **then**

    $\left(t_a^{k+1}, t_b^{k+1}\right) \leftarrow \left(\lambda^k, \mu^k\right)$

  else

    $c^{\lambda^k} \leftarrow \frac{\bar{a}_m}{\lambda} + \bar{b}_m \lambda$

    $c^{\mu^k} \leftarrow \frac{\bar{a}_m}{\mu} + \bar{b}_m \mu$

    for $j = 1$ to m **do**

      $z_j^{\lambda^k} \leftarrow Z\left(f_j, d_j, \sigma_j, \lambda^k\right) \qquad z_j^{\mu^k} \leftarrow Z\left(f_j, d_j, \sigma_j, \mu^k\right)$

      $g_j^{\lambda^k} \leftarrow \begin{cases} h_j z_j^{\lambda^k} \sigma_j & \text{if } z_j^{\lambda^k} > 0 \\ \frac{1}{2} h_j z_j^{\lambda^k} \sigma_j & \text{otherwise} \end{cases} \qquad g_j^{\mu^k} \leftarrow \begin{cases} h_j z_j^{\mu^k} \sigma_j & \text{if } z_j^{\mu^k} > 0 \\ \frac{1}{2} h_j z_j^{\mu^k} \sigma_j & \text{otherwise} \end{cases}$

      $c^{\lambda^k} \leftarrow c^{\lambda^k} + g_j^{\lambda^k} \sqrt{\lambda^k} \qquad c^{\mu^k} \leftarrow c^{\mu^k} + g_j^{\mu^k} \sqrt{\mu^k}$

    end for

    $\left(t_a^{k+1}, t_b^{k+1}\right) \leftarrow \begin{cases} \left(\lambda^k, z_b^k\right) & \text{if } c_\lambda^k > c_\mu^k \\ \left(z_a^k, \mu^k\right) & \text{otherwise} \end{cases}$

  end if

end for

$t_m^\star = (t_a^{k+1} + t_b^{k+1})/2$

$c_m^\star = \frac{\bar{a}_m}{t_m^\star} + \bar{b}_m t_m^\star$

for $j = 1$ to m **do**

  $z_j^\star \leftarrow Z\left(f_j, d_j, \sigma_j, t_m^\star\right)$

  $g_j^\star$ at $z_j^\star, \sigma_j,$ and h_j

  $c_m^\star \leftarrow c_m^\star + g_j^\star \sqrt{t_m^\star}$

end for

return $t_m^\star$ and $c_m^\star$

Algorithm 5 Single-Family Roundoff Algorithm

Data
$(t^\star, \mathbf{y}^\star)$
Initialize
$\gamma \leftarrow t^\star$ //Begin Roundoff Procedure
for $j = 1$ to n **do**
  Find t^j and $\pi^j \in \mathbb{Z}_+ \ni y_j = t^j 2^{\pi^j}$ and $t^j \in [\gamma, 2\gamma)$
end for
Add a subscript h to t^j and $\pi^j \ni t^j_h \leq t^j_{h+1}$
for $\phi = 1$ to n **do**
  $\pi^j_\phi \leftarrow \begin{cases} \pi^j_h - 1 & \text{if } h \leq \phi \\ \pi^j_h & \text{otherwise} \end{cases}$
  $k^j_\phi \leftarrow 2^{\pi^\phi_j}$
  for $j = 1$ to n **do**
    $z^\phi_j \leftarrow Z\left(f_j, d_j, \sigma_j, t_j\right)$
  end for
  $t^\star_\phi = \underset{t_\phi}{\text{argmin}} = c\left(t^\star_\phi, k_\phi\right)$ using the pr-MF-JRP Algorithm, Step 4.
end for

IX. Evaluation

We tested the effectiveness of our solution procedures with a computational study. We chose the time unit to be one day. The ranges of values for other problem data were chosen to reflect realistic industrial situations. The detailed data and results are available from the authors. The data were generated from uniformly distributed parameters as shown in Table 2.

The results of the test runs are in Table 3. The table lists performance measures, evaluating the performance of the our algorithm against (1) the lower bound determined by Problem M_R, and (2) a traditional JRP solution approach, a procedure that first solves the deterministic version of the problem and then computes fill-rate safety stock requirements. Problems were generated for each of the following problems:

- the (single-family) JRP-FR,
- the MFJRP-FR with 2 families,
- the MFJRP-FR with 5 families, and
- the MFJRP-FR with 10 families,

In the single-family environment, our algorithm typically resulted in a solution that averaged 1.01% above the lower bound, with cost averaging 0.06% below the the traditional approach. For the MFJRP-FR with 2 families, our algorithm averaged 0.61% above the lower bound and 0.10% below the traditional approach. For the MFJRP-FR with 5 families, our algorithm averaged 1.01% above the lower bound and 4.72% below the traditional approach. For the MFJRP-FR with 10 families, our algorithm averaged 0.98% above the lower bound and 4.09% below the traditional approach. Relative to the lower bound, our algorithmï£¡s performance predictably worsens as the number of families increases, but improvements over the traditional approach show greater improvements for problems with larger families.

X. Summary

We have extended the Joint Replenishment Problem to a multi-family environment that explicitly includes the consideration of safety stocks for a fill

Algorithm 6 Multi-Family Roundoff Algorithm

Data
$(t^\star, \mathbf{y}^\star)$
Initialize
$\gamma \leftarrow t^\star$ //Begin Roundoff Procedure
for $(i,j) \in \mathbb{N} \times \mathbb{N}$ **do**
  Find t^{ij} and $\pi^{ij} \in \mathbb{Z}_+ \ni y_{ij} = t^{ij} 2^{\pi^{ij}}$ and $t^{ij} \in [\gamma, 2\gamma)$
end for
Add a subscript h to t^{ij} and $\pi^{ij} \ni t^{ij}_h \leq t^{ij}_{h+1}$
for $\phi = 1$ to n^2 **do**
  $\pi^{ij}_\phi \leftarrow \begin{cases} \pi^{ij}_h - 1 & \text{if } h \leq \phi \\ \pi^{ij}_h & \text{otherwise} \end{cases}$
  $k^{ij}_\phi \leftarrow 2^{\pi^\phi_{ij}}$
  for $j = 1$ to n **do**
    $z^\phi_{ij} \leftarrow Z\left(f_{ij}, d_{ij}, \sigma_{ij}, t_{ij}\right)$
  end for
  $t^\star_\phi = \underset{t_\phi}{\text{argmin}} = c\left(t^\star_\phi, k_\phi\right)$ using the pr-MF-JRP Algorithm, Step 4
end for

rate criterion. We exploited a number of properties of the problem to develop an efficient solution procedure. Our procedure affords the practitioner an approach that specifies the length of replenishment cycles as well as required working stock and safetey stock levels whose costs, along with the fixed ordering costs, are minimized.

The form of our solution is a basic period cyclic schedule, with item multipliers restricted to integer-powers-of-two. The integer-powers-of-two solution form greately eases the pratitioner's task of scheduling and coordinating deliveries. We do, however, need to develop procedures to take the solution for our problem, $(T, \mathbf{K}, \mathbf{k})$, and actually construct a feasible schedule of deliveries to each of the basic periods in a cycle to ensure that constraints on receiving and handling incoming shipments are met.

A parametric analysis of the model is needed to determine the sensitivity of the solution to changes in the cost function as relevant costs and product demand are subject to frequent revisions.

Table 2: Problem Sampling Parameters

Parameter	Distribution
Family order cost (\$)	$U\,(200, 500)$
Delivery lead-time (days)	$U\,(0, 3)$
Item order cost (\$)	$U\,(75, 150)$
Item holding cost (\$/unit/day)	$U\,(0.08, 0.2)$
Item demand mean (units/day)	$\lfloor U\,(50, 500) \rfloor$
Item demand standard deviation (% of demand mean)	$U\,(0.25, 0.5)$
Item fill-rate (% of demand met from inventory)	$U\,(0.96, 0.999)$

Table 3: Performance of the MFJRP-FR Algorithm

		% above LB				% Below Alg. 2			
Model	**n**	**Avg.**	σ	**Min**	**Max**	**Avg.**	σ	**Min**	**Max**
JRP-FR	50	0.22	0.21	0.00	0.69	0.06	0.08	0.00	0.40
MFJRP-FR(2)	45	0.61	0.48	0.06	2.11	0.10	0.25	0	1.27
MFJRP-FR(5)	50	1.01	0.44	0.18	2.54	4.72	2.94	0.03	10.45
MFJRP-FR(10)	50	0.98	0.27	0.41	1.74	4.09	1.45	0.79	7.22

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