Coordinating Shipments from Multiple Supplier Locations in a Capacitated Staging Environment

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The periodic-review Multiple-Family JRP (pr-MF-JRP) extends the JRP to an environment with multiple replenishment locations and a constrained unloading capacity at the stocking point. The mathematical model's objective is to minimize family fixed costs, item fixed costs, inventory holding costs for both cycle and safety stocks, and overtime costs for the unloading operations at the stocking location. The proposed heuristic provides good lower bounds and improved performance over existing approaches.

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I. INTRODUCTION

The JRP has been studied extensively under a wide variety of assumptions. The original deterministic JRP addresses a stocking location, such as a retailer, who must replenish several products from a single supplier in a time-stationary demand environment. Economies exist when the retailer replenishes several items together. The items that originate from a single supplier form a single family of items. A fixed, major setup cost, which we will call the family order cost, is associated with the order. A major component of the family fixed cost is the transportation cost from the supplier to the stocking location. An additional fixed minor setup cost, which we will call the item order cost, is assessed for every item included in the order. Item costs included processing fees charged by the supplier and costs associated with reviewing inventory levels performing administrative tasks for each item that is ordered. A holding costs is assessed for average inventory levels of each item held at the stocking location. The stochastic JRP additionally considers the holding cost of safety stock held at the stocking location to hedge against demand uncertainty. The deterministic JRP can be modified to address stochastic demand with the inclusion of safety stock costs in the objective function (Eynan & Kropp, 1997).

Typically, retailers and wholesalers have multiple replenishment sources, each represented by a family of items. The JRP with multiple families has not been significantly studied in the literature. Transportation costs are determined by the supplier location as well as its distance from the stocking location. As such, fixed ordering costs are different for each family of items. The importance of considering the multiple family environment is the need to coordinate the deliveries from all of the suppliers to a stocking location that has limited capacity to stage deliveries. The consideration of a capacity constraint to account for unloading time presents a development that is different from the machine-type constraint found in related literature. Overtime and overtime costs can be considered in the formulation of our problem and in the implementation of the joint replenishment when it is economical. Another difference between the unloading capacity constraint and the machine capacity constraint is that our problem may have multiple unloading constraints, one per family. In practice this could play out as a company receiving daily shipments from different suppliers while only allocating the morning hours to unloading and stocking. The afternoon hours would be used for staging outgoing shipments.

This article further modifies Eynan and Kropp's model (1997) to account for multiple stocking locations and a capacitated staging area where shipments are unloaded.

The pr-MF-JRP is developed under the following assumptions.

- Normal i.i.d. demand with known parameters
- Multiple product families, each family representing a supplier
- Replenishment to stock
- Adequate safety stock levels to meet specified service levels
- A capacity on the time available to unload the delivered items with the option of additional labor and associated labor costs
- Relevant costs include:
  - Major ordering cost for each family
  - Minor ordering cost for each product within the family
  - Cycle inventory holding costs
  - Safety stock inventory holding costs

In §II., we survey relevant JRP literature. In §III., we present the multi-family JRP with capacity constraints and discuss the problem's properties. We propose a solution process in §IV.. Examples are provided in §V. A computational study is presented in §VI. followed by concluding remarks are in §VII..

II. REVIEW OF THE LITERATURE

The majority of the JRP Literature addresses the single-family deterministic variant, where demand is known and constant. Examples of Deterministic JRP include research by Goyal (1974), Andres & Emmons (1976), Joneja (1989), Federgruen & Zheng (1992), Fung & Ma (2001), and Viswanathan & Ma (2002). Robinson & Lawrence (2004) add a production capacity constraint to their JRP model. Examples of research addressing the stochastic JRP include Balintfy (1964), who initiated the research stream on the (s, c, S) continuous review control rule, denoting the *should order level* (s), the *can order level* (c), and the order-up-to level (S). For an item i, $S_i - s_i$ units are ordered when the item's inventory reaches s_i units. When the item's inventory level is below c_i and an order is placed for other items in the same family, $S_i - c_i$ units are ordered. See Federgruen et al. (1984) for solution algorithms.

The periodic review approach was shown to dominate continuous review approaches (Atkins & Iyogun, 1988; Eynan & Kropp, 1997). A more recent stream of research, e.g. Eynan & Kropp (2007) and Tagaras & Vlachos (2002), investigates the periodic review approach to the stochastic JRP with safety stock held to hedge against stockouts.

For a small cost penalty, integer powers of two policies afford greater ease in scheduling compared to integer-only policies (See Jackson et al., Jackson et al. (1985)). Roundy (1985) shows that the upper-bound cost penalty of a policy with integer powers of two is at most 2% above the continuous solution. In a stochastic-demand environment, safety stock costs contribute to the integer powers of two cost penalty. In the case of flexible basic periods, Karalli & Flowers (2006) observed the cost penalty to average 5.35% when the objective function additionally included family setup costs and safety stock costs.

III. THE MULTI-FAMILY PROBLEM

We propose a modification and an extension to the algorithm by Eynan & Kropp (1997), allowing for multiple families and one or more capacity constraints. The periodic review Multiple Family JRP (pr-MF-JRP) is a continuous-time, infinite-horizon extension of the JRP where n items are ordered from m families, or supply locations.

The pr-MF-JRP allows deliveries to be staggered so that receiving and unloading is more manageable to the retailer. For our problem, we assume one replenishment source per unloading period. For the family, there is a major order cost and a delivery lead-time. The demand for each item is i.i.d., normally distributed with known mean and standard deviation. We further assume that items are neither complements to nor substitutes for the other items. For each item, there is a known, constant unloading and stocking rate, a minor order cost, and a specified service level that must be met by maintaining safety stock to cover the delivery lead-time and the replenishment cycle. Backlogging is not allowed.

The following notation will be employed in this section:

i	subscript denoting the i^{th} family; $i = 1, \ldots, m$
j	subscript denoting the j^{th} item; $j = 1, \ldots, n$
(i, j)	notation for item i in family j
m	number of families
n	number of items
$\mathbb{M}$	set of family subscripts; $\mathbb{M} = \{1, \ldots, m\}$
$\mathbb{N}$	set of item subscripts; $\mathbb{N} = \{1, \ldots, n\}$
$\mathbb{P}$	set of integer powers of two; $\mathbb{P} = \{2^p : p \in \mathbb{W}\}$
$\mathbb{W}$	set of whole numbers; $\mathbb{W} = \{0, \ldots, \infty\}$

The input parameters into Problem F are:

U	total available unloading time
U_i	maximum available unloading time for family i
L_i	delivery lead-time for family i
d_{ij}	demand mean for item (i, j)
σ_{ij}	demand standard deviation for item (i, j)
z_{ij}	standard normal variate corresponding to the service level required for item (i, j)
p_{ij}	unloading rate for item $(i.j)$
ρ_{ij}	unloading capacity factor for (i, j); $\rho_{ij} = \frac{d_{ij}}{p_{ij}}$

The decision variable of Problem F are:

T	length of basic period
ω_i	overtime needed to unload shipment of family i
ω	$= \sum_{i \in \mathbb{M}} \omega_i$
K_i	multiplier for family i
k_{ij}	multiplier for item (i, j)
T_i	$= TK_i$
T_{ij}	$= TK_i k_{ij}$
b_{ij}	$= \frac{1}{2} h_{ij} d_{ij}$
g_{ij}	$= z_{ij} \sigma_{ij} h_{ij}$
$\mathbf{K}$	$m-$vector of K_is
$\mathbf{k}$	$m \times n-$matrix of k_{ij}s
$\mathbf{k}_{i\cdot}$	i^{th} row of $\mathbf{k}$
$\boldsymbol{\omega}$	$m-$vector of ω_is

Problem F . Find $(T, \boldsymbol{\omega}, \mathbf{K}, \mathbf{k})$ so as to:

Minimize
$$C\left(T, \boldsymbol{\omega}, \mathbf{K}, \mathbf{k}\right) = \sum_{i \in \mathbb{M}} \left(\frac{A_i}{T_i} + \varsigma_i \, \omega_i \right) + \sum_{i \in \mathbb{M}} \sum_{i \in \mathbb{N}} \left(\frac{a_{ij}}{T_{ij}} + b_{ij} T_{ij} + g_{ij} \sqrt{T_{ij}} \right) \tag{1}$$

Subject to
$$U_i + \omega_i - T_i \sum_{i \in \mathbf{N}} \rho_{ij} k_{ij} \geq 0; \, \forall i \in \mathbb{M} \tag{2}$$
$$T > 0 \tag{3}$$
$$K_i \in \mathbb{P}; \quad i \in \mathbb{M} \tag{4}$$
$$k_{ij} \in \mathbb{P} \times \mathbb{P}; \, \forall \left(i, j\right) \in \mathbb{M} \times \mathbb{N} \tag{5}$$
$$\boldsymbol{\omega} \geq 0 \tag{6}$$

The necessary feasibility condition, constraint (2), assumes a separate delivery time for, and therefore a unique unloading time constraint on each product family. Problem F is separable into m single-family problems. However this is only the case for the continuous relaxation of Problem F if the firm takes delivery of all families at once, such as at a cross-dock. For a such a scenario (2) is replaced with (7).

$$U + \omega - T \sum_{i \in \mathbb{M}} K_i \sum_{j \in \mathbb{N}} \rho_{ij} k_{ij} \geq 0 \tag{7}$$

3.1 Solution Properties

Karalli & Flowers (2006) show that a triplet $(T, \mathbf{K}, \mathbf{k})$ that solves their MFELSP-SS, a multi-family extension of the ELSP, can always be represented in anchor form (AF). This is a convenient solution property that affords an efficient solution procedure to only seek solutions in AF. which can be adapted to the JRP multi-family problem. It computes solutions in anchor form (AF), a representation of $(T, \mathbf{K}, \mathbf{k})$ whose properties are listed in Definition 1 below. With some modifications to account for the delivery lead-time, the algorithm can be employed to solve Problem F.

Definition 1. A solution, $(T, \boldsymbol{\omega}, \mathbf{K}, \mathbf{k})$ is in AF $\Leftrightarrow$ the following conditions hold:

1. $0 < T \in \mathbb{R}$
2. $\mathbf{K} = [1, K_2, \ldots, K_n] \in \mathbb{P}^m$ with $1 \leq K_2 \leq \cdots \leq K_n$
3. $\mathbf{k}_{i\cdot} = [1, k_{i1}, \ldots, k_{in}] \in \mathbb{P}^n$ with $1 \leq k_{i1} \leq \cdots \leq k_{in}, \forall i \in \mathbb{N}$

3.2 Continuous Relaxation

A common approach in JRP algorithm development is to compare the cost of the solution to a lower bound. With the removal of the unloading constraints (2) and the following variable substitutions, we formulate Problem R, the continuous relaxation of Problem F. After substituting the variables as shown constraints (4) and (5) are removed.

A superscript v is added to y_{ij}. At $v = 0$ the single-item procedure computes the solution to the deterministic counterpart, which treats

Variable Substitutions:

x_i	$= TK_i$
y_{ij}	$= TK_i k_{ij}$
λ_{ij}	dual variable for Problem R
$\mathbf{x}$	$m-$vector of x_is
$\mathbf{y}$	$m \times n-$matrix of y_{ij}s
$\mathbf{y}_{i\cdot}$	i^{th} row of $\mathbf{y}$

Problem R
Find $\mathbf{x}$ & $\mathbf{y}$ so as to:

Minimize
$$C(\mathbf{x}, \mathbf{y}) = \sum_{i \in \mathbf{M}} \left(\frac{A_i}{x_i} \right) + \sum_{i \in \mathbb{M}} \sum_{j \in \mathbf{N}} \left(\frac{a_{ij}}{y_{ij}} + b_{ij} y_{ij} + g_{ij} \sqrt{y_{ij}} \right) \tag{8}$$

Subject to
$$x_i - y_{ij} \leq 0; \ \forall (i, j) \in \mathbb{M} \times \mathbb{N} \tag{9}$$
$$\mathbf{x}, \mathbf{y} \geq 0 \tag{10}$$

Problem R can be separated into m single family problems, Problem R_1 to Problem R_m. To establish the existence of a solution for Problem R_i, we next derive its KKT conditions.

KKT Conditions for Problem R_i Given the objective function (8), $C : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$, and constraints (9) and (10), all continuously differentiable everywhere, $\forall i \in \mathbb{M}, j \in \mathbb{N}, \exists\, 0 \leq \lambda_{ij} \in \mathbb{R}$ such that (9) and (10) hold, and

Complementarity Condition

$$\lambda_{ij}\left(x_i - y_{ij}\right) = 0; \qquad \forall\left(i,j\right) \qquad (11)$$

Gradient Conditions

$$-\frac{A_i}{x_i^2} + \sum_{j\in\mathbb{N}} \lambda_{ij} = 0 \qquad (12)$$

$$-\frac{a_{ij}}{y_{ij}^2} + b_{ij} + \frac{1}{2}\frac{g_{ij}}{\sqrt{L_i + y_{ij}}} - \lambda_{ij} = 0 \qquad (13)$$

Proposition 2. *Let* $(x_i, \mathbf{y}_{i\cdot})$ *be a KKT point in Problem* F_i. *There exists* $\mu = 1,\ldots,m$, *such that* $x_i = y_{i1} = \cdots = y_{i\mu} \leq y_{i,\mu+1} \leq \cdots \leq y_{in}$.

Proof. The gradient condition (12) implies that $\exists\, \mu = 1,\ldots,m$ such that $\lambda_{i\mu} > 0$, which in turn implies $x_i - y_{i\mu} = 0$ in condition (11). It then follows that $x_i = y_{i1} = \cdots = y_{i\mu} \leq y_{i,\mu+1} \leq \cdots \leq y_{in}$. □

IV. SOLUTION

The proposed solution will produce a cyclic schedule consisting of the basic period length, family multipliers, item multipliers, and required overtime that give the lowest total fixed costs, inventory carrying costs, and overtime costs. The family multipliers and item multipliers are integer powers of two so as to facilitate the scheduling process. The stocking location will receive a delivery from supplier i every $T \times K_i$ weeks. Item (i, j) will be shipped every $T \times K_i \times k_{ij}$ weeks with the shipment of family i.

The solution to the pr-MF-JRP proceeds in four steps:

Step 1. *The Item Step*. Find the optimal replenishment for each item.

Step 2. *The Family Step*. Find the KKT points for each Family and select the cost minimizer.

Step 3. *The Roundoff Step*. Roundoff to integer powers of two.

Step 4. *The Overtime Step*. Assess the required overtime for each trial solution in Step 3 and compute the total cost.

The foregoing steps can also be used in the traditional, single-family JRP by setting $m = 1$ before proceeding.

The item step computes the optimal replenishment times for each item without considering family fixed costs. The values generated in this step are only used in the ensuing family step, where the family fixed costs are considered. The family step serve two purposes. First, the total cost computed using the cost minimizers in the family step is a lower bound, which is used to evaluate the performance of the algorithm in §VI.. Second, the cost minimizers are the starting point for the roundoff step.

The roundoff step prepares $m \times n$ candidate solutions. Each solution in this step consists of the basic period length, family multipliers, and item multipliers. With the addition of the required overtime to each candidate solution in step 4, the algorithm selects the candidate solution that minimizes the cost function (1).

4.1 The Item Step

A procedure given in Eynan & Kropp (2007) is employed (see the pr-MF-JRP Algorithm, Step 1) to find $y_{ij}^*, \forall\,(i,j) \in \mathbb{M} \times \mathbb{N}$.

Additional Notation for Step 1:

ϵ — convergence tolerance; $\varepsilon = 0.001$

v — superscript indication the v^{th} iteration

indicator variables

$$\delta = \begin{cases} 1 & \text{if } T\sum_{i\in\mathbb{M}} K_i \sum_{j\in\mathbb{N}} \rho_{ij} k_{ij} \geq U \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_i = \begin{cases} 1 & \text{if } T_i \sum_{i\in\mathbb{N}} \rho_{ij} k_{ij} \geq U_i \\ 0 & \text{otherwise} \end{cases}$$

A superscript v is added to y_{ij}. At $v = 0$ the single-item procedure computes the solution to the deterministic counterpart, which treats the item's demand as constant. At iteration $v + 1$, the algorithm computes y_{ij}^{v+1} as shown in statement 4 of the pr-MF-JRP Algorithm, Step 1.

The sequence $\{y_{ij}\}^v$ is shown to converge to y_{ij}^* in Eynan & Kropp (2007), where computational experiments demonstrate that it only takes a few iterations (two or three) for y_{ij}^v to approach y_{ij}^*. At the end of Step 1, we have solutions for n^2 items.

The pr-MF-JRP Algorithm, Step 1 Item Step

Require:
$v = 0$
1: **Initialize** $y_{ij}^0 = \sqrt{\frac{a_{ij}}{b_{ij}}}$
2: **while** $\left| y_{ij}^{v+1} - y_{ij}^v \right| > \epsilon$ **do**
3: $v \leftarrow v + 1$
4: $y_{ij}^{v+1} = \sqrt{\dfrac{a_{ij}}{b_{ij} + \frac{g_{ij}}{2\sqrt{L_i + y_{ij}^v}}}}$
5: **end while**
6: **return** y_{ij}^{v+1}

4.2 The Family Step

For each family $i = 1, \ldots, m$, locate the KKT points and select $c_i^\star$ the one that minimizes c_i^μ in (14), $\forall\, \mu = 1, \ldots, n$. Compute $C\left(\mathbf{x}^\star, \mathbf{y}^\star\right)$ using (15).

$$c_i^\mu \left(x_i^\mu, \mathbf{y}_{i\cdot}^\mu\right) = \frac{A_i}{x_i} + \sum_{j \in \mathbb{N}} \left(\frac{a_{ij}}{y_{ij}} + b_{ij} y_{ij} + g_{ij} \sqrt{y_{ij}} \right) \tag{14}$$

$$C\left(\mathbf{x}^\star, \mathbf{y}^\star\right) = \sum_{i \in \mathbb{M}} c_i^\star \tag{15}$$

This step is given in the pr-MF-JRP Algorithm, Step 2.

Proposition 3. *$\forall\, \mu = 1, \ldots, n,\ \exists\, \lambda_{ij} \ni\ x_i^\mu = y_{i1}^\mu = \cdots = y_{i\mu}^\mu$ and $x_i^\mu < y_{i,\mu+1}^\mu, \cdots, y_{in}^\mu \iff \left(x_i^\mu, y_{i,\mu+1}^\mu, \ldots, y_{in}^\mu\right)$ produced by the pr-MF-JRP Algorithm, Step 2 is a KKT point for Problem R_i.*

Proof. It must be shown that if $x_i^\mu = y_{i1}^\mu = \cdots = y_{i\mu}^\mu$ and (Case 1) if $x_i^\mu < y_{i,\mu+1}^\mu, \cdots, y_{in}^\mu$, the KKT conditions (11), (12), and (13) are satisfied, and (Case 2) if $\exists\, j = \mu + 1, \ldots, n, \ni\ x_i^\mu \geq y_{ij}^\mu$, then $\left(x_i^\mu, y_{i,\mu+1}^\mu, \ldots, y_{in}^\mu\right)$ is not a KKT point.

Case 1. Problem R_i is separated into μ problems, Problems R_i^μ, $\mu = 1, \ldots, n$. The objective function of Problem R_i^μ, for some μ, can be rewritten as (16).

$$c_i^\mu \left(x_i^\mu, y_{i,\mu+1}, \ldots, y_{in}\right) = c_i^\mu \left(x_i^\mu\right) + c_i^\mu \left(y_{i,\mu+1}\right) + \cdots + c_i^\mu \left(y_{in}\right) \tag{16}$$

$$c_i^\mu \left(x_i^\mu\right) = \frac{A_i}{x_i^\mu} + \sum_{j=1}^{\mu} \left(\frac{a_{ij}}{x_i^\mu} + b_{ij} x_i^\mu + g_{ij} \sqrt{L_i + x_i^\mu} \right) \tag{17}$$

$$c_i^{\mu}(y_{ij}) = \frac{a_{ij}}{y_{ij}^{\mu}} + b_{ij}y_{ij}^{\mu} + g_{ij}\sqrt{L_i + y_{ij}^{\mu}} \qquad j = \mu+1,\ldots,n \tag{18}$$

For items $(i,j), j \leq \mu$, whose cost is (17), the value of λ_{ij} is constructed in (19). The first order condition (FOC) for (17) is given in (20).

For each item $(i,j), \ni j > \mu$, whose cost is (18), $\lambda_{ij} = 0$ by construction. With $\lambda_{ij} = 0$, condition (13) reduces to the FOC of the item's cost function given in (21), which is solved with the pr-MF-JRP Algorithm, Step 1. Conditions (11) and (13) are met for $j \geq \mu$.

$$-\frac{a_{ij}}{y_{ij}^2} + b_{ij} + \frac{1}{2}\frac{g_{ij}}{\sqrt{L_i + y_{ij}}} = \lambda_{ij}; \qquad j = 1,\ldots,\mu \tag{19}$$

$$-\frac{A_i}{x_i^2} + \sum_{j=1}^{\mu}\left(-\frac{a_{ij}}{y_{ij}^2} + b_{ij} + \frac{1}{2}\frac{g_{ij}}{\sqrt{L_i + y_{ij}}}\right) = 0 \tag{20}$$

$$-\frac{a_{ij}}{y_{ij}^2} + b_{ij} + \frac{1}{2}\frac{g_{ij}}{\sqrt{L_i + y_{ij}}} = 0; \qquad j = \mu+1,\ldots,n \tag{21}$$

Condition (11) is satisfied $\forall\, j \in \mathbb{N}$ by construction. Taken together, (19) and $\lambda_{ij} = 0$ for $j > \mu \iff$ (13). Replacing the summation expression in (21) with the LHS of (20) results in condition (12).

For items $(i,j), j \leq \mu$, whose cost are in the second term of (16), condition (11) implies that at least one of the $\lambda_{ij} \geq 0$ (by Proposition 2). The FOC for the first and second terms of (16) is given in (20). Therefore condition (12) is satisfied. Condition (13) is satisfied as well.

Case 2. Let $x_i^{\mu} > y_{i\underline{j}}$ for some $\underline{j} > \mu$, constraint (9) does not hold for $\underline{j}$. The unsatisfied constraint precludes $\left(x_i^{\mu}, y_{i,\mu+1}^{\mu}, \ldots, y_{i\underline{j}}^{\mu}, \ldots, y_{in}^{\mu}\right)$ from being a KKT point. □

Proposition 4. *The solution* $(\mathbf{x}^{\star}, \mathbf{y}^{\star})$ *produced by the family step is a lower bound for problem F*

Proof. By Proposition 2 the algorithm always produces an optimal solution. By (15) $C(\mathbf{x}^{\star}, \mathbf{y}^{\star})$ the lowest cost KKT points for each family $i \in \mathbb{M}$. During the rounding step, recovering T, $\mathbf{K}$ and $\mathbf{k}$ using the variable substitution equations, and solving for T results in the objective value $C(T^{\star}, \mathbf{K}^{\star}, \mathbf{k}^{\star})$ in (1) being greater than $C(\mathbf{x}^{\star}, \mathbf{y}^{\star})$. □

4.3 The Roundoff Step

The roundoff step of the pr-MF-JRP Algorithm, Step 3 follows the procedure given in Karalli & Flowers (2006).

4.4 The Overtime Step

Once the $m \times n$ trial solution have been prepared, the pr-MF-JRP Algorithm, Step 4 finds the value of T that minimizes the objective function. *The Overtime Step* takes into account unloading capacity requirements by changing the value of the indicator variable δ_i for family i so that the overtime cost is in the objective function for that shipment when overtime is used.

The pr-MF-JRP Algorithm, Step 2 The Family Step

Initialization: re-index $\mathbf{y}$ so that $y_{i1} \le y_{i2} \le y_{in}$
for $\mu \in \mathbb{N}$ **do**
  $v = 0;\ x_i^0 = \sqrt{\frac{A_i + \sum_{j=1}^{\mu} a_{ij}}{\sum_{j=1}^{\mu} b_{ij}}}$
  while $\left| y_{ij}^{v+1} - y_{ij}^{v} \right| > \epsilon$ **do**
    $v \leftarrow v + 1;\ x_i^{v+1} = \sqrt{\frac{A_i + \sum_{j=1}^{\mu} a_{ij}}{\sum_{j=1}^{\mu} b_{ij} + \frac{\sum_{j=1}^{\mu} g_{ij}}{2\sqrt{L_i + x_i^v}}}}$
  end while
  if $x_i^{v+1} \le y_{i\mu}$ **then** //This is a KKT point
    $y_{i1}^{\mu}, \ldots, y_{i\mu}^{\mu} \leftarrow x_i^{v+1};\quad y_{i,\mu+1}^{\mu} \leftarrow y_{i,\mu+1}, \ldots, y_{in}^{\mu} \leftarrow y_{in};\quad \mathbf{y}_{i\cdot}^{\mu} \leftarrow \left[y_{i1}^{\mu}, \ldots, y_{i\mu}^{\mu}, y_{i,\mu+1}^{\mu}, \ldots, y_{in}^{\mu} \right]$
    $c_i^{\mu}\left(x_i^{\mu}, \mathbf{y}_{i\cdot}^{\mu}\right) \leftarrow \frac{A_i}{x_i^{\mu}} + \sum_{j=1}^{n} \left(\frac{a_{ij}}{y_{ij}^{\mu}} + b_{ij} y_{ij}^{\mu} + g_{ij} \sqrt{L_i + y_{ij}^{\mu}} \right)$
  else
    $c_i^{\mu}\left(x_i^{\mu}, \mathbf{y}_{i\cdot}^{\mu}\right) \leftarrow \infty$
  end if
end for
return $c_i^{\star} \leftarrow \min\left\{ c_i^{\mu} | \mu \in \mathbb{N} \right\}$

V. EXAMPLES

5.1 The Single-Family JRP with Safety Stocks and Overtime

Family Data

$A = \$264 \qquad c = \$1,000 \qquad L = 0.118 \qquad U = 0.064$

Item Data

i	a_i	d_i	p_i	σ_i	h_i	z_i
1	\$137	429	11,709	376.40	\$1.14	2.08
2	91	335	12,179	264.38	0.10	3.43
3	82	455	11,973	377.29	0.90	2.18
4	167	360	11,459	236.70	0.75	3.24
5	120	459	12,932	309.18	0.09	2.36

Solution

The solution algorithm proceeds in three steps: (a) the item step, (b) the family step, and (c) the round-off step.

(a) The Item Step

The items re-indexed so that $x_1 \le x_2 \le \cdots \le x_5$. The item step yields the following results:

$x_1 = 0.327 \qquad x_2 = 0.373 \qquad x_3 = 0.398$
$x_4 = 0.592 \qquad x_5 = 1.623$

The pr-MF-JRP Algorithm, Step 3 The Roundoff Step

Initialization:

$\gamma \leftarrow \min\{y_{ij} : (i,j) \in \mathbb{M} \times \mathbb{N}\}$

for $(i,j) \in \mathbb{M} \times \mathbb{N}$ **do**

  $y_{ij} \leftarrow t_{ij} 2^{\pi_{ij}} \ni t_{ij} \in [\gamma, 2\gamma) \ \& \ \pi_{ij} \in \mathbb{W}$

end for

Add a superscript, h, to each π_{ij} and create the set $\Pi = \left\{\pi_{ij}^h : t_{ij}^h \leq t_{ij}^{h+1}, h = \{1, \ldots, m\cdot\}\right\}$

for $\phi = 1, \ldots, m \cdot n$ **do**

  for $h = 1, \ldots, m \cdot n$ **do**

    $\pi_{ij}^{\phi} \leftarrow \begin{cases} \pi_{ij}^h - 1 & \text{if } \phi \leq h \\ \pi_{ij}^h & \text{otherwise} \end{cases}; \qquad k_{ij}^{\phi} \leftarrow \pi_{ij}^{\phi}$

  end for

  for $i = 1, \ldots, m$ **do**

    $K_i^{\phi} \leftarrow \min_j \left\{k_{ij}^{\phi} : j = 1, \ldots, n\right\}$

  end for

end for

The pr-MF-JRP Algorithm, Step 4 The Overtime Step

for $\phi = 1, \ldots, m \cdot n$ **do**

  $v = 0; \qquad T^v = \sqrt{\frac{2\sum_{i\in\mathbb{M}} \frac{A_i}{K_i} + \sum_{i\in\mathbb{M}}\sum_{j\in\mathbb{N}} a_{ij}}{\sum_{i\in\mathbb{M}}\sum_{j\in\mathbb{N}} b_{ij}}}$

  if $T^v K_i \sum_{j\in\mathbb{N}} \rho_{ij} k_{ij} > U_i$ **then** //overtime needed to unload shipment

    $\delta_i = 1$

  end if

  while $\left|y_{ij}^{v+1} - y_{ij}^{v}\right| > \epsilon$ **do**

    $v \leftarrow v + 1; \qquad T^{v+1} = \sqrt{\frac{\sum_{i\in\mathbb{M}}\left(\frac{A_i}{K_i} + \sum_{j\in\mathbb{N}} \frac{a_{ij}}{K_i k_{ij}}\right)}{\sum_{i\in\mathbb{M}}\sum_{j\in\mathbb{N}}\left(b_{ij} K_i k_{ij} + \delta_i \omega_i \rho_{ij} K_i k_{ij} + \frac{g_{ij}}{2\sqrt{L_i + T^v K_i k_{ij}}}\right)}}$

    if $T^{v+1} K_i \sum_{j\in\mathbb{N}} \rho_{ij} k_{ij} > U_i$ **then** //overtime needed to unload shipment

      $\delta_i = 1$

    else //no overtime needed

      $\delta_i = 0$

    end if

  end while

  $T^{\phi} \leftarrow T^{v+1}$

end for

$T^{\phi\star} \leftarrow \underset{T^{\phi}}{\text{argmin}}\, C\left(T^{\phi}, \mathbf{K}^{\phi}, \mathbf{k}^{\phi}\right)$

Table 1: Performance of the Single-Family JRP-SS-OT Algorithm

m	y	x_1	x_2	x_3	x_4	x_5	$C(y, \mathbf{x})$
1	0.754	0.754	0.373	0.398	0.592	1.623	*infeasible*
2	0.815	0.815	0.815	0.398	0.592	1.623	*infeasible*
3	0.624	0.624	0.624	0.398	0.592	1.623	*infeasible*
4	0.617	0.617	0.617	0.617	0.592	1.623	$3,743.95
5*	0.662	0.662	0.662	0.662	0.662	1.623	$3,807.89

(b) The Family Step
The results for each iteration m are displayed in Table 1.
The solution to the continuous relaxation, with the original indices, is:

$$y = 0.662 \qquad x_1 = 0.662$$
$$x_2 = 0.662 \qquad x_3 = 0.662$$
$$x_4 = 0.662 \qquad x_5 = 1.623$$
$$C(y, \mathbf{x}) = 3,743.95 \text{ (our lower bound)}$$

(c) The Round-off & Overtime Steps
The results for each iteration f of the Round-off step are given in the Table 2. $T = 0.598$

Table 2: Results of the Round-off & Overtime Steps of the Single-Family Example

f	T	k_1	k_2	k_3	k_4	k_5	ω	$C(T, \omega, \mathbf{k})$
1	0.471	1	2	2	2	4	0.112	$4,112.77
2	0.494	1	1	2	2	4	0.106	4,178.03
3	0.525	1	1	1	2	4	0.097	3,944.66
4	0.570	1	1	1	1	4	0.093	3,850.85
5*	0.598	1	1	1	1	2	0.058	3,809.55

$\omega = 0.058$
$\mathbf{k} = \{1, 1, 1, 1, 2\}$
$C(y, \mathbf{x}) = 3,743.95$ (our lower bound) The objective value resulting from the algorithm is $3,809.55, which is %1.75 larger than the lower bound of $3,743.95.

5.2 The pr-MF-JRP with Safety Stocks and Overtime

The following example is for a three-family problem. Each family has three items. The inputs into the problem follow:
Family Data

i	A_i	L_i	U_i
1	$312	0.112	0.094
2	249	0.085	0.064
3	278	0.100	0.062

Item Data

i	j	a_{ij}	d_{ij}	p_{ij}	σ_{ij}	h_{ij}	z_{ij}
1	1	$85	373	11,681	0.8499	$0.11	1.55
1	2	81	628	10,994	0.6807	0.26	1.44
1	3	122	340	10,055	0.6638	1.15	2.05
2	1	83	871	12,066	0.8727	1.10	1.85
2	2	139	697	11,685	0.6652	0.70	1.69
2	3	87	464	11,407	0.7345	0.03	1.77
3	1	139	792	13,316	0.8507	1.23	1.72
3	2	99	342	12,850	0.8921	0.16	2.03
3	3	102	766	10,633	0.8525	1.07	1.72

Solution The solution to the continuous relaxation, with the original indexes, is:

$$\begin{array}{lll} y_1 = 0.913 & y_2 = 0.467 & y_3 = 0.440 \\ x_{11} = 1.415 & x_{21} = 0.467 & x_{31} = 0.440 \\ x_{12} = 0.913 & x_{22} = 0.481 & x_{32} = 1.182 \\ x_{13} = 0.913 & x_{23} = 2.648 & x_{33} = 0.440 \end{array}$$
$$C(y, \mathbf{x}) = 8,391.99 \text{ (our lower bound)}$$

The results for each iteration f of the Round-off & Overtime steps are given in the Table 3.

The solution to the problem, with the original indices, is:

$$\mathbf{K} = \begin{bmatrix} K_1 = 2 & K_2 = 1 & K_3 = 1 \end{bmatrix}$$
$$\mathbf{k} = \begin{bmatrix} k_{11} = 1 & k_{21} = 1 & k_{31} = 1 \\ k_{12} = 1 & k_{22} = 1 & k_{32} = 2 \\ k_{13} = 1 & k_{23} = 4 & k_{33} = 1 \end{bmatrix}$$

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_1 = 0.021 & \omega_2 = 0.073 & \omega_3 = 0.024 \end{bmatrix}$$

$$C(T, \boldsymbol{\omega}, \mathbf{K}, \mathbf{k}) = \$8,712.64$$

Table 3: Results of the Round-off & Overtime Steps of the pr-MF-JRP Example

f	T	i	K_i	k_{i1}	k_{i2}	k_{i3}	ω_i	$C(T, \boldsymbol{\omega}, \mathbf{K}, \mathbf{k})$
1	0.31	1	2	1	1	1	0.000	\$9,396.80
		2	2	1	1	4	0.131	
		3	1	1	4	2	0.041	
2	0.33	1	2	1	1	1	0.000	9,134.11
		2	2	1	1	4	0.148	
		3	1	1	4	1	0.024	
3	0.38	1	2	1	1	1	0.010	9,046.92
		2	1	1	2	8	0.154	
		3	1	1	4	1	0.039	
4	0.40	1	2	1	1	1	0.017	8,960.75
		2	1	1	1	8	0.143	
		3	1	1	4	1	0.056	
5	0.41	1	2	1	1	1	0.020	8,908.95
		2	1	1	1	8	0.147	
		3	1	1	2	1	0.023	
6*	0.42	1	2	1	1	1	0.021	8,712.64
		2	1	1	1	4	0.073	
		3	1	1	2	1	0.024	
7	0.46	1	1	2	1	2	0.002	9,064.23
		2	1	1	1	4	0.086	
		3	1	1	2	1	0.032	
8	0.50	1	1	2	1	1	0.000	8,977.98
		2	1	1	1	4	0.095	
		3	1	1	2	1	0.038	
9	0.51	1	1	1	1	1	0.000	9,043.84
		2	1	1	1	4	0.098	
		3	1	1	2	1	0.040	

VI. COMPUTATIONAL ANALYSIS

The effectiveness of the proposed scheduling solution procedure is tested with a computational study. Problem instances were generated for the 3×3 (three Families, each with three items) pr-MF-JRP with different overtime costs. The time unit was chosen to be one week, as it is a common production period in industry. The ranges of values for other problem data were similarly chosen to reflect realistic industrial situations. The detailed data and results are available from the authors. The data were generated from uniformly distributed parameters as shown in Table 4 below.

The results of the pr-MF-JRP test runs are in Table 5. The table lists performance measures, evaluating the performance of the pr-MF-JRP Algorithm against (1) the lower bound determined by Problem R, and (2) a traditional JRP solution approach, a polynomial time algorithm that first solves the deterministic version of the problem and then computes safety stock and overtime requirements .

For the pr-MF-JRP Algorithm with OT costs averaging \$1,000, the total cost average is 1.11% above the lower bound and 3.82% below the traditional approach. Relative to the lower bound, our algorithm's performance predictably worsens as overtime cost increases, but improvements over the traditional polynomial time algorithm increases. Fig. 1 below graphs the performance of our algorithm as overtime cost increases relative to the lower bound as well as the polynomial algorithm.

Table 4: Problem Sampling Parameters

Parameter	Distribution
Family order cost ($)	$U\,(200, 350)$
Item order cost ($)	$U\,(75, 150)$
Item holding cost ($/unit/week)	$U\,(0.01, 0.50)$
Item demand mean (units/week)	$U\,(250, 1000)$
Item demand standard deviation (proportion of demand mean)	$U\,(0.6, 0.9)$
Item unloading rate (units/week)	$U\,(10,000, 15,000)$
Item service level (probability all demand is met)	$U\,(0.90, 0.9999)$
Overtime cost ($/week)	Varied*
Unloading capacity (weeks)	$U\,(0.05, 0.075)$
Lead time (weeks)	$U\,(0.08, 0.12)$

*Separate runs were conducted with overtime costs sampled from: $U\,(750, 1250)$, $U\,(1750, 2250)$, $U\,(2750, 3250)$, $U\,(3750, 4250)$, $U\,(4750, 5250)$, $U\,(5750, 6250)$, $U\,(6750, 7250)$, $U\,(7750, 8250)$

Table 5: Performance of the pr-MF-JRP Algorithm

	% above LB				% Below Alg. 2			
Model	**Avg.**	σ	**Min**	**Max**	**Avg.**	σ	**Min**	**Max**
OT(1000)	1.11	0.51	0.43	2.28	2.68	2.07	0.22	9.73
OT(2000)	2.30	1.18	1.00	5.24	4.65	2.74	1.02	11.42
OT(3000)	3.06	1.51	0.95	6.69	5.30	3.09	0.84	11.89
OT(4000)	3.61	1.60	1.64	9.56	4.63	2.19	0.89	8.83
OT(5000)	5.65	3.09	1.18	16.14	7.47	5.56	1.82	22.65
OT(6000)	5.98	2.92	1.66	12.37	7.16	2.81	2.31	11.47
OT(7000)	7.76	3.80	2.23	18.43	7.58	3.72	2.11	17.34
OT(8000)	8.33	4.66	2.69	19.74	6.76	4.65	2.40	25.14

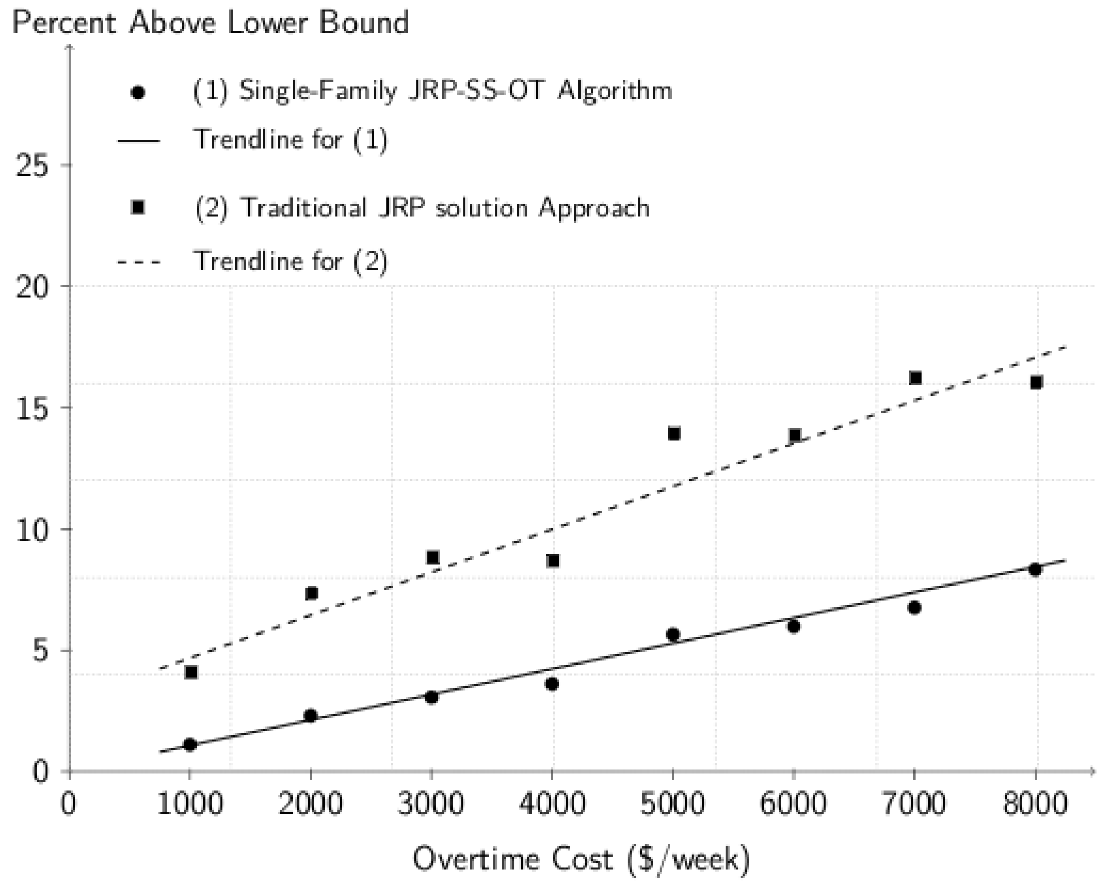

Figure 1: Performance of the pr-MF-JRP Algorithm with respect to the lower bound and a traditional JRP solution approach.

VII. SUMMARY

In this paper, the JRP was extended to include multiple replenishment locations, safety stock costs, and unloading capacity constraints with overtime costs in the optimization. The form of our solution is a basic period cyclic schedule, with item multipliers restricted to integer-powers-of-two. Problem properties were exploited to develop a polynomial run-time solution procedure. Results show that the practitioner would benefit from using this procedure over the traditional alternative that likely represents current approaches that ignore the cost of carrying the safety stock as well as staging capacity when planning replenishment policies.

There are a few avenues for future research on the pr-MF-JRP. Procedures are required that take $(T, \boldsymbol{\omega}, \mathbf{K}, \mathbf{k})$ and schedule the arrivals in such a way as to use as much of the facility's available capacity. Such procedures would prevent multiple arrivals on the same day while allowing other days to be idle. Another problem is the dynamic problem of adjusting order quantities to account for actual sales in an uncertain demand environment and resultant inventory levels expected by the end of the cycle. The popularity of fill rates as a measure of customer service in distribution environments justifies the need to develop scheduling methodology that explicitly considers fill rates safety stock.

The difficulty in accurately estimating model parameters such as fixed and variable costs, the demand for each item, and the lead time for each product family calls for additional computational studies. A parametric analysis of the model is needed to determine the sensitivity of the solution to changes in the cost function resulting from revisions in the values of the parameters.

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