Locating Service Facilities with Concave Variable Costs

Robert Aboolian*
California State University at San Marcos, San Marcos, California, USA

Emilio Carrizosa
Universidad de Sevilla, Sevilla, Spain

Vanesa Guerrero
Universidad de Sevilla, Sevilla, Spain

Jiamin Wang
Long Island University Post Campus, Brookville, New York, USA

We consider a nonlinear version of the Uncapacitated Facility Location Problem (UFLP). The total cost in consideration consists of a fixed cost to open facilities, a travel cost in proportion to the distance between demand and the assigned facility, and an operational cost at each open facility, which is assumed to be a concave nondecreasing function of the demand served. Thus we call the problem Uncapacitated Facility Location Problem with Concave Operating Cost (UFLPCOC). Specifically, we assume that service facilities are to be located and customers seek service from the closest open facility. As a consequence, an explicit constraint is needed in the model to impose closest assignment. An exact solution approach, which is called the Search and Cut algorithm, is presented. This approach is mainly based on sequentially improving the lower and upper bounds for UFLPCOC. Lower bounds are obtained by solving a UFLP model with extra linear constraints. To find an upper bound, we present a heuristic that is based on a neighborhood search procedure starting from the solution to a mixed integer programming model. An approximation solution approach is also suggested that explores linear approximation to transform the model into a mixed integer linear programming problem. Computational results are presented. It is found that the cost structure has a significant effect on intractability of the problem and that the Search and Cut algorithm dominates the approximation solution approach in general.

---

* Corresponding Author. E-mail address: raboolia@csusm.edu

I. INTRODUCTION

The uncapacitated facility location problem (*UFLP*) on a network is a classical model in the field of Discrete Location Theory (Brandeau and Chiu, 1989). It is also called the simple plant location problem in the literature, though plant location does not exclude applications of the model in the service industry. In its basic formulation, facilities are located and demand regions are allocated to these facilities with an objective to minimize the total cost, including the setup cost to open facilities and the transportation cost from each demand region to its assigned facility. Krarup and Pruzan (1983) proved that the *UFLP* problem is NP-hard. Different from the median, center and maximal covering models, the number of facilities to set up is not necessary to specify in advance for *UFLP*. Instead, it is a by-product of optimization through balancing the fixed cost and the transportation cost. The problem is uncapacitated as there is no limit on demand allocated to each open facility. Efroymson and Ray (1966) noted that the basic formulation of *UFLP* exhibits both the closest assignment and single assignment properties, i.e., at the optimal solution, each demand node is fully served by the closest open facility.

A branch and bound algorithm was developed in Efroymson and Ray (1966) for the *UFLP* problem. In the procedure, the linear relaxation problem is solved to yield an initial lower bound. A branch-and-bound tree is built by continuously branching from a non-integer location decision variable in the current solution and a series of linear relaxation problems are then generated and solved to improve the lower bound progressively until all location decision variables in the current solution become all integers. Cornuejols, Fisher and Nemhauser (1977) presented a Lagrangian relaxation algorithm for the *UFLP* problem and derived the worst-case bound of a greedy heuristic. Morris (1978) reported that solving the linear relaxation problem of a tighter basic *UFLP* formulation (William, 1974) returned the exact optimal solution to the original model in 96% of the computational instances.

The reader is referred to Krarup and Pruzan (1983) and Verter (2011) for literature reviews on the *UFLP* problem and its capacitated counterpart. A major extension of *UFLP* is to include the variable or operational costs in the objective function. A facility's variable cost is a function of the overall demand it serves. The presence of economies of scale apparently justifies a concave variable cost as the unit operational cost normally decreases in the number of units processed. Verter and Dincer (1995) noted that the closest assignment property no longer holds for the *UFLP* problem with concave variable costs. Piecewise concave variable costs were considered in Efroymson and Ray (1966), while variables costs were modeled by power functions in Khumawala and Kelly (1974). Efroymson and Ray (1966) suggested an approach to transform *UFLP* with piecewise linear variable costs into an expanded *UFLP* formulation that associates a potential facility with each line segment. Soland (1974) developed a branch and bound algorithm for general concave variable costs. The procedure treats *UFLP* as a fixed-charge transportation problem and obtains the optimal solution to each sub-problem in a linear programming form via simple inspection. Each sub-problem is generated by branching on a selected demand node to add an additional line segment to approximate the associated variable cost function. The location and capacity acquisition model Verter and Dincer (1995) formulated is very close to the model studied in the current study. However, in their formulation, the closest assignment property is not enforced. The authors devised a branch and bound algorithm, in which the branching step is similar to the one Soland (1974) proposed, but the resulting sub-problems are still *UFLP* models.

In this study, we consider locating service facilities to satisfy customer demand. In particular, customers travel to the immobile facilities to seek service. There is a rich literature in location theory on coveragetype problems where capacity and congestion at each service facility are modelled through a queuing system. We refer the reader to Berman and Krass (2015) for a recent literature review. But in the current study, we are mainly concerned with minimizing the overall cost for locating and operating such service facilities in addition to customers' travel expenses and disregard any uncertainty in demand or supply. Although the service provider incurs the fixed and variable cost of the facilities while the customers incur the traveling cost, our objective is to find a socially optimal solution that minimizes the overall cost. Consequently, we study a *UFLP* problem in nature.

We note such a problem is suitable for facilities such as supermarkets and postal offices with relatively stable and homogenous demand where the capacity can be determined easily so as to keep the respective service time and waiting time constant or negligible relative to the amount of time customers spend traveling to facilities. Although a service facility location problem is considered in this study, we note that the model and solution methods developed in this paper can be easily extended to many more applications including the location of manufacturing facilities, distribution centers, or online retail warehouses where the shipping cost is paid by the facilities.

We assume that the variable cost at each facility is a concave function of the number of customers served. Though the closest assignment property would not carry over to the *UFLP* model with variable costs, it was not enforced in any of the existing formulations. That is, customers may be assigned to a distant facility even though an open facility is nearby. Such a feature might be feasible in a centralized decision-making environment, but it would not work in a customer-choice situation unless some mechanism could force customers to follow the assignments. As noted in Berman, Krass and Wang (2016), models that ignore customer choices may significantly underestimate the demand at some facilities. In our study, a constraint is applied to ensure that customers be assigned to the closest facility that is open.

The model presented in the current study works in a distributed decision-making environment where customers seek for service at a facility at their own choice. As such a decision-making environment reflects the reality of a majority of the service providers, the model will enable the management to predict the relevant costs more accurately and make a robust facility location decision. The model will also enable the management to make trade-offs among different cost components, including the facilities' fixed cost and variable cost as well as customers' transportation cost.

The problem of interest, which we call the uncapacitated facility location problem with concave operational costs (*UFLPCOC*) is modeled in Section 2. A motivating example is analyzed to show that closest assignment property does not hold automatically. An approximation solution approach and an exact solution approach are presented in Section 3. The approximation approach first approximates each variable cost function as a concave piecewise linear function and then solves the resulting MIP model. The exact solution approach is mainly based on obtaining efficient lower and upper bounds for the problem. Lower bounds are obtained by solving the MIP model with additional linear constraints. To find an upper bound, we present a heuristic that is based on a neighborhood search heuristic over the location sets optimal to the MIP model. This exact solution approach adopts the search-and-cut methodology proposed by Aboolian, Cui and Shen (2012). Computational results are presented in Section 4. Finally, some conclusions and a suggestion on future research topics are provided in Section 5.

II. PROBLEM STATEMENT

Let $N(|N| = n)$ be the set of customer demand regions and $M(|M| = m)$ be the set of candidate locations for the facilities. We denote the demand rate at node $i \in N$ by λ_i. If a facility is located at site j, we call it facility j. For facility $j \in M$, denote by $f_j \geq 0$ the fixed cost to open the facility and by $F_j(\Lambda_j)$ the variable cost to operate with demand Λ_j. Owing to economies of scale, we assume that $F_j(\Lambda_j)$ is a concave function in Λ_j. Let c_{ij} be the access cost of a customer from node i to facility j. It is assumed that c_{ij} is proportional to the distance between node i and facility j. It follows that $c_{ii} = 0$ and $c_{ij} = c_{ji}$ hold for any i and j.

We use $S \subseteq M$ to denote the set of open facilities. Define $i[S] \in S$ to be the facility in S that serves customers residing at node $i \in N$. As explained in the previous section, customers travel to the closest open facility for service. It follows $i[S] = \underset{j\in S}{\operatorname{argmin}}\{c_{ij}\}$ holds for node $i \in N$. Define $E_j(S)$ to be the set of demand nodes served by facility $j \in S$. Note $E_j(S) = \{i \in N: j = i[S]\}$. Given the definitions, it is easy to derive $\Lambda_j = \sum_{i \in E_j(S)} \lambda_i$.

Define $G(S)$ to be the total cost under a given location set S, i.e.,

$$G(S) = \sum_{j \in S} [\, f_j + F_j(\sum_{i \in E_j(S)} \lambda_i)] + \sum_{j \in S} \sum_{i \in E_j(S)} \lambda_i \, c_{ij}.$$

The Uncapacitated Facility Location Problem with Concave Operating Cost ($UFLPCOC$) is formulated as:

$$\min_{S \subseteq M} \{G(S)\}.$$

We now write the problem as a nonlinear integer program. Let x_j be a binary variable, which is one if a facility opens at site j and zero otherwise. Let y_{ij} be a binary variable, which is one if customers at node i travel to facility j for service and zero otherwise. Given the above definitions, $UFLPCOC$ is formulated as:

$$\min \sum_{j \in M} \left[f_j x_j + F_j \left(\sum_{i \in N} \lambda_i y_{ij} \right) \right] + \sum_{i \in N} \sum_{j \in M} \lambda_i c_{ij} y_{ij}$$

s.t.

$$\sum_{j \in M} y_{ij} = 1, \qquad \forall i \in N, \qquad (1)$$
$$y_{ij} \leq x_j, \qquad \forall i \in N, j \in M, \qquad (2)$$
$$\sum_{l \in M} c_{il} y_{il} \leq (c_{ij} - L) x_j + L, \qquad \forall i \in N,\ j \in M, \qquad (3)$$
$$x_j,\ y_{ij} \in \{0, 1\}, \forall i \in N, j \in M. \qquad (4)$$

In the model, constraints (1) ensure that every demand node is served, while constraints (2) enforce customers travel to open facilities only. Constraints (3) require that each demand node is assigned to the open facility with the least access cost. In constraints (3), L is a positive number sufficiently large (e.g. $L = \max_{j \in M, i \in N}\{c_{ij}\}$).

If function $F_j() = 0$ at any site j, then constraints (3) become redundant as minimizing the access cost implies the closest assignment property. In addition, the binary constraint on y_{ij} can be replaced by constraint $0 \leq y_{ij} \leq 1$ because the single assignment property also holds. However, if function $F_j()$ is not all zero, constraints are needed to impose closest assignment and the binary constraint on y_{ij} cannot be relaxed.

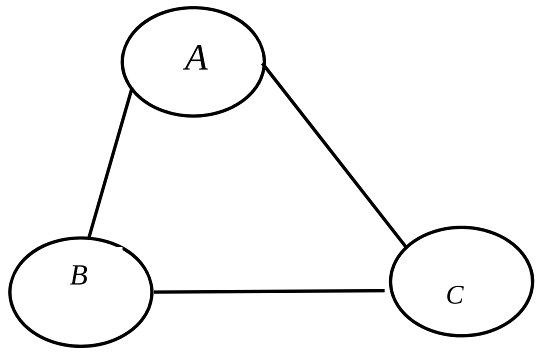

FIGURE 1. A NETWORK OF THREE NODES

To motivate the study, consider the network presented in Fig. 1. The demand weights originated from nodes A, B and C are $\lambda_A = \lambda_C = 3$ and $\lambda_B = 4$. The fixed cost to locate a facility at each of the three nodes is $f_A = 5$, $f_B = 12$ and $f_C = 6$. The variable cost to operate an open facility at each node is in the form of a power function $F_j(\Lambda_j) = a_j \Lambda_j^{b_j}$, with $a_A = a_C = 3$, $a_B = 4$ $b_A = 0.5$, and $b_B = b_C = 1$. In addition, we assume $c_{AB} = 5$, $c_{BC} = 4$ and $c_{AC} = 6$. All possible solutions are presented in Table 1.

We note that the optimal solution to the above model is to locate a facility at node A (solution 1 in Table 1). The total cost under this solution is 52.5. However, solution 6, i.e., locating facilities at node A and C while assigning node B to the facility at node A (instead of node C that is closer) would be more economical with a total cost of 47.9 if the closest assignment requirement were not imposed.

TABLE 1. SOLUTIONS TO THE MOTIVATING EXAMPLE

Solution index	Location	Allocation	Fixed cost	Variable cost	Access cost	Total cost
1	$x_A = 1$	$y_{AA} = y_{BA} = y_{CA} = 1$	5	9.5	38	52.5
2	$x_B = 1$	$y_{AB} = y_{BB} = y_{CA} = 1$	12	40	27	79
3	$x_C = 1$	$y_{AC} = y_{BC} = y_{CC} = 1$	6	30	34	70
4	$x_A = x_B = 1$	$y_{AA} = y_{BB} = y_{CA} = 1$	17	23.3	18	58.3
5	$x_A = x_B = 1$	$y_{AA} = y_{BB} = y_{CB} = 1$	17	33.2	12	62.2
6	$x_A = x_C = 1$	$y_{AA} = y_{BA} = y_{CC} = 1$	11	16.9	20	47.9
7	$x_A = x_C = 1$	$y_{AA} = y_{BC} = y_{CC} = 1$	11	26.2	16	53.2
8	$x_A = x_B = x_C = 1$	$y_{AA} = y_{BB} = y_{CC} = 1$	23	30.2	0	53.2

The fixed cost and the variable cost are increasing while the access cost is decreasing in the number of open facilities. Examining the solutions in Table 1, we observe that it may be better off to open more facilities because the reduction in the access cost could offset the increment in the other two cost components. In particular, the rise in the variable cost could be controlled via a prudent allocation of customer demand to facilities. If, however, customers choose which facility to patron at will, a solution may turn out to be costly because the actual variable cost was underestimated. For instance, solution 6 in Table 1 results in a variable cost of 16.9 when customers from node B are assigned to the facility at node A. If all customers patron the closest facilities, i.e., customers from node B seek service at the facility at node C, the solution's variable cost will jump to 26.2, which will make this solution sub-optimal.

A variant of the $UFLPCOC$ model is to remove the access cost $\sum_{i \in N} \sum_{j \in M} \lambda_i c_{ij} y_{ij}$ from the objective function and impose $\sum_{i \in N} \sum_{j \in M} \lambda_i c_{ij} y_{ij} \leq D$ as a constraint, where D is a constant. This is equivalent to minimizing the sum of the fixed cost to open the facilities and the variable cost to operate the facilities subject to an additional constraint that bounds the total access cost from above. It is evident that this new model optimizes the location decision from the perspective of the service provider, instead of the system that includes the service facilities and the customers.

Let's re-examine Table 1. Note that solution 4 and solution 6 are not feasible under the closest-assignment constraint anyway. It is easy to see for the new model, solution 8, solution 5, solution 7, solution 3 or solution 1 will be optimal, respectively, if $D < 12$, $12 \leq D < 16$, $16 \leq D < 34$, $34 \leq D < 38$, or $D \geq 38$. Coincidently, the original formulation and the new formulation with a sufficiently large D value have an identical optimal solution in this example.

The new formulation is similar to the original one in the sense of making a trade-off among the facilities' fixed cost and the variable cost as well as the customers' access cost. If D is small, solutions with high access costs tend to be infeasible, and therefore it is optimal to open more facilities. If D is large, there are more feasible solutions and therefore it is optimal to open fewer facilities.

As will be shown later, the original model is computationally intractable due to the closest-assignment constraint and the non-linear variable cost functions, not the total access cost function, which is linear. As a result, the computational complexity of solving the two formulations should be about the same. We thus focus on the original model in this study.

Note that $UFLPCOC$, a nonlinear integer program, is intractable in nature. The closest assignment constraints make the branch and bound algorithms developed by Soland (1974) and Verter and Dincer (1995) not applicable any more. To efficiently solve this problem, we propose an exact solution approach in the next section.

III. ALGORITHMS FOR UFLPCOC

Similar to Aboolian, Cui and Shen (2012), our exact solution approach is mainly based on obtaining efficient lower and upper bounds for $UFLPCOC$. In Section 3.1, we develop an MIP, for which the optimal objective function value is a lower bound to $UFLPCOC$. In Section 3.2, we present a heuristic, which is based on neighborhood search over the location set optimal to the MIP. The heuristic is used to find an upper bound to $UFLPCOC$. The exact approach presented in Section 3.3 is based on successive lower and upper bound improvements for $UFLPCOC$.

3.1 A Lower Bound for $UFLPCOC$

In order to determine a lower bound for $UFLPCOC$, concave function $F_j(\Lambda_j)$ is replaced with a piecewise linear function $\underline{F}_j(\Lambda_j)$ for any $j \in M$ with break points $\Lambda_j^{\min} = \Lambda_{j0} < \Lambda_{j1} < \Lambda_{j2} < \cdots < \Lambda_{j,K_j-1} < \Lambda_{j,K_j} = \Lambda_j^{\max}$, where K_j is the number of line segments, and $\underline{F}_j(\Lambda_j) \leq F_j(\Lambda_j)$ is ensured for $\Lambda_j^{\min} \leq \Lambda_j \leq \Lambda_j^{\max}$ with the equality reached at the break points only. Let p_{jk} and q_{jk} be, respectively, the slope and y-intercept of line segment $k = 1, 2, \cdots, K_j$. As depicted in Fig.

2, segment k is bounded by $\Lambda_{j,k-1}$ and Λ_{jk}. Because $F_j(\Lambda_j)$ is concave and increasing, $p_{j1} > p_{j2} > \cdots > p_{jK}$ and $q_{j1} < q_{j2} < \cdots < q_{jK}$ follow. The lemma below can be proven easily.

Lemma 1. $\underline{F}_j(\Lambda_j) = p_{jk}\Lambda_j + q_{jk} = \min_{1\le s\le K}\{p_{js}\Lambda_j + q_{js}\}$ holds for $\Lambda_{j,k-1} \le \Lambda_j \le \Lambda_{jk}$.

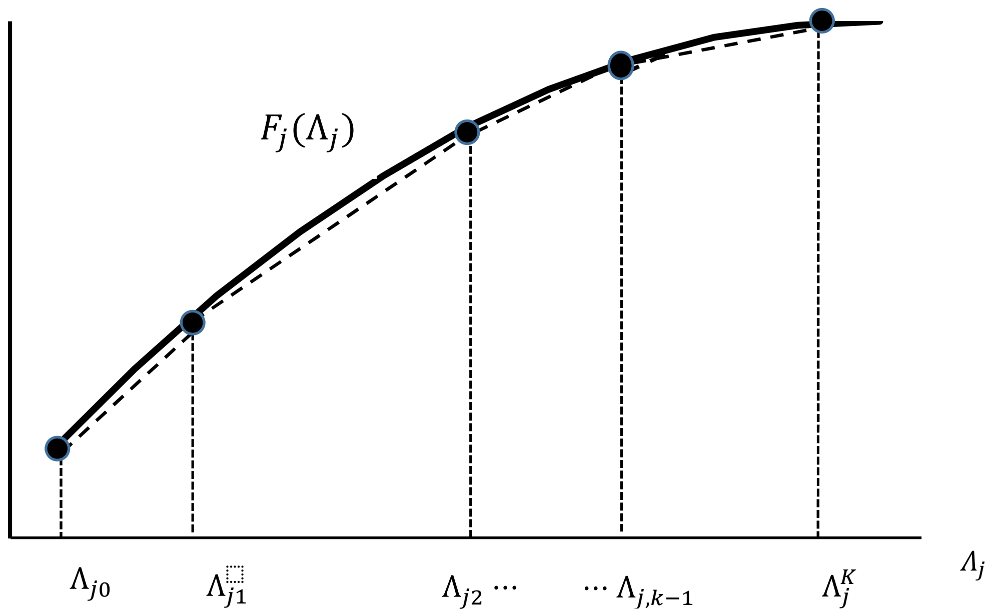

FIGURE 2. PIECEWISE LINEAR APPROXIMATION

If facility j is open, then it can absorb all demand. We hence define $\Lambda_j^{\max} = \sum_{i\in N}\lambda_i$. On the other hand, the minimum demand that an open facility j can fulfill is $\Lambda_j^{\min} = \sum_{\{i\in N: c_{ij} < c_{ik}, \forall k\in M\}}\lambda_i$, namely the total demand weights from the nodes closer to site j than to any other. Note $\Lambda_j^{\min} = \lambda_j$ if M and N are identical. It is evident that the following formulas return the slope and y-intercept of line segment k once the break points are determined,

$$p_{jk} = \frac{F_j(\Lambda_{jk}) - F_j(\Lambda_{j,k-1})}{\Lambda_{jk} - \Lambda_{j,k-1}}$$

and

$$q_{jk} = F_j(\Lambda_{jk}) - p_{jk}\Lambda_{jk}.$$

Given the number of line segments K_j, we examine two approaches to select break points. In the first approach, the break points are evenly spaced between $\Lambda_j^{\min}$ and $\Lambda_j^{\max}$. That is, $\Lambda_{jk} = \Lambda_j^{\min} + \frac{k(\Lambda_j^{\max} - \Lambda_j^{\min})}{K_j}$ for $k = 0, 1, \cdots, K_j$. In the second approach, an iterative procedure is adopted to continuously reduce the maximum relative error between the original function $F_j(\Lambda_j)$ and its approximate $\underline{F}_j(\Lambda_j)$ until the gap is small enough or after a pre-chosen number of iterations.

The relative error function over each line segment k can be re-written as $R_k(\Lambda_j) = \frac{F_j(\Lambda_j) - (p_{jk}\Lambda_{jk} + q_{jk})}{F_j(\Lambda_j)}$ for $\Lambda_{j,k-1} \le \Lambda_j \le \Lambda_{jk}$. If $F_j(\Lambda_j)$ is derivable, its first-order derivative can be formulated as $R_k'(\Lambda_j) = \frac{(p_{jk}\Lambda_{jk} + q_{jk})F'(\Lambda_j) - p_{jk}F_j(\Lambda_j)}{F_j^2(\Lambda_j)}$. The next lemma suggests an important property of function $R_k(\Lambda_j)$.

Lemma 2. Function $R_k(\Lambda_j)$ is unimodal in $\Lambda_j \in [\Lambda_j^{k-1}, \Lambda_j^k]$.

Proof. Let $u(\Lambda_j) = (p_{jk}\Lambda_{jk} + q_{jk})F'(\Lambda_j) - p_{jk}F_j(\Lambda_j)$. Its first-order derivative is $u'(\Lambda_j) = (p_{jk}\Lambda_{jk} + q_{jk})F''(\Lambda_j)$. Recall that $F_j(\Lambda_j)$ is concave and hence $F_j''(\Lambda_j) < 0$. Consequently, $u'(\Lambda_j) < 0$ follows and $u(\Lambda_j)$ is monotone decreasing. Note that $F_j^2(\Lambda_j)$ is increasing, we thus know that $R_k'(\Lambda_j)$ is monotone decreasing. It is easy to verify that $R_k'(\Lambda_{j,k-1}) > 0$ and $R_k'(\Lambda_{jk}) < 0$. It follows that there exists $\overline{\Lambda}_{jk} \in [\Lambda_{j,k-1}, \Lambda_{jk}]$ such that $R_k'(\overline{\Lambda}_j^k) = 0$.

In light of the lemma above, we can apply the bisection method to find the unique point $\overline{\Lambda}_{jk}$ on line segment k , which maximizes the relative error. We next present a procedure to implement the second approach.

- Step-1. **Let $\boldsymbol{\Lambda_{jk} = \Lambda_j^{\min} + \frac{k(\Lambda_j^{\max} - \Lambda_j^{\min})}{K_j}}$ for $\boldsymbol{k = 0, 1, \cdots, K_j}$. Let $\boldsymbol{t = 0}$.**
- Step-2. **For each line segment $\boldsymbol{k}$, compute the slope $\boldsymbol{p_{jk}}$ and the $\boldsymbol{y}$-intercept $\boldsymbol{q_{jk}}$, then apply the bisection method to find $\boldsymbol{\overline{\Lambda}_{jk}}$ that solves equation $\boldsymbol{(p_{jk}\Lambda_{jk} + q_{jk})F'(\Lambda_j) = p_{jk}F_j(\Lambda_j)}$.**
- Step-3. **Let $\boldsymbol{k_j^{\max} = \arg\max_{k=1,2,\cdots,K_j}\{R_k(\overline{\Lambda}_{jk})\}}$, $\boldsymbol{k_j^{\min} = \arg\min_{k=1,2,\cdots,K_j}\{R_k(\overline{\Lambda}_{jk})\}}$ and $\boldsymbol{t = t+1}$. If $\boldsymbol{\max_{k=1,2,\cdots,K}\{R_k(\overline{\Lambda}_{jk})\} - \min_{k=1,2,\cdots,K}\{R_k(\overline{\Lambda}_{jk})\} \le \varepsilon}$ or $\boldsymbol{t}$ exceeds a pre-selected level, stop.**
- Step-4. **If $\boldsymbol{k_j^{\max} = K}$, then go to Step 5. Otherwise, let $\boldsymbol{\Lambda_{j,k_j^{\max}} = \Lambda_{j,k_j^{\max}} - \xi}$, and go to Step 2.**
- Step-5. **Let $\boldsymbol{\Lambda_{j,k_j^{\max-1}} = \Lambda_{j,k_j^{\max-1}} + \xi}$. Go to Step 2.**

In the above procedure, ε is the expected bound of the gap and ξ is step size. At each iteration, either break point of the line segment with the highest relative error is adjusted to reduce the gap.

By replacing $F_j(\Lambda_j)$ with $\underline{F_j}(\Lambda_j)$ in $UFLPCOC$, we establish the following mixed-integer linear program, which we call $LBMIP$:

$$\min \sum_{j\in M} f_j x_j + \sum_{j\in M} V_j + \sum_{i\in N}\sum_{j\in M} \lambda_i c_{ij} y_{ij}$$

s.t.

(1), (2), (3), (4)

$$V_j + (1 - z_{jk})B \ge \textstyle\sum_{i\in N} \lambda_i y_{ij} p_{jk} + q_{jk}, \qquad \forall j \in M, 1 \le k \le K, \tag{5}$$

$$\textstyle\sum_{1\le k\le K} z_{jk} \ge x_j, \qquad \forall\, j \in M, \tag{6}$$

$$0 \le z_{jk} \le 1, \qquad \forall\, j \in M,\ 1 \le k \le K\,. \tag{7}$$

where B is a number sufficiently large. In the above model, constraints (5) and (6) ensure that the operational cost V_j is zero if no facility is open at site j, while V_j equals $\underline{F_j}(\Lambda_j)$ when otherwise. Applying Lemma 1, we note that a decision variable $z_{jk} = 1$ if and only if $x_j = 1$ and $\Lambda_j = \sum_{i\in N} \lambda_i\, y_{ij}$ lies on segment k, while $z_{jk} = 0$ otherwise.

Alternatively, we can associate a pseudo-facility with each line segment of function $\underline{F_j}(\Lambda_j)$ and develop a formulation equivalent to model $LBMIP$ as follows.

$$\min \sum_{j\in M}\sum_{1\le k\le K} (f_j + q_{jk})X_{jk} + \sum_{i\in N}\sum_{j\in M}\sum_{1\le k\le K} \lambda_i(c_{ij} + p_{jk})\, Y_{ijk}$$

s.t.

$$\textstyle\sum_{j\in M}\sum_{1\le k\le K} Y_{ijk} = 1, \qquad \forall i \in N,$$

$$Y_{ijk} \le X_{jk}, \qquad \forall i \in N,\ j \in M, 1 \le k \le K,$$

$$\textstyle\sum_{l\in M}\sum_{1\le k\le K} c_{il} Y_{ilk} \le (c_{ij} - L)\sum_{1\le k\le K} X_{jk} + L, \qquad \forall i \in N,\ j \in M,$$

$$X_{jk}, Y_{ijk} \in \{0,1\}, \qquad \forall i \in N,\ j \in M,\ 1 \leq k \leq K.$$

where decision variable $X_{jk} = 1$ if facility j is open and Λ_j lies on segment k of function $\underline{F_j}(\Lambda_j)$, while decision variable $Y_{ijk} = 1$ if customers from node i seek service from facility j and Λ_j lies on segment k of function $\underline{F_j}(\Lambda_j)$. The model above has a linear objective function as in the original $UFLP$ problem. However, because the objective coefficients include q_{jk} and p_{jk}, the closest assignment constraint is necessary. Note that this model requires more decision variables and constraints, we therefore use model $LBMIP$ in our computational experiment described in the next section. The theorem below shows that model $LBMIP$ returns a lower bound to the $UFLPCOC$ model.

Theorem 1. Denote by $\underline{Z}^*_{LBMIP}$ and $Z^*_{UFLPCOC}$, respectively, the optimal objective value of model $LBMIP$ and model $UFLPCOC$. $\underline{Z}^*_{LBMIP}$ is a lower bound of $Z^*_{UFLPCOC}$, i.e.

$$\underline{Z}^*_{LBMIP} \leq Z^*_{UFLPCOC}.$$

Proof. Let $S^*_{UFLPCOC}$ be the optimal location set to model $UFLPCOC$. We note that by construction, solution $S^*_{UFLPCOC}$ can be converted into a feasible solution to model $LBMIP$. Let $\underline{Z}$ be the $LBMIP$ objective value of this feasible solution. Since $\underline{F_j}(\Lambda_j) \leq F_j(\Lambda_j)$, it is easy to derive $Z^*_{UFLPCOC} \geq \underline{Z} \geq \underline{Z}^*_{LBMIP}$.

3.2 An Upper Bound for $UFLPCOC$

We note that when function $F_j()$ is linear for any $j \in M$, $UFLPCOC$ reduces to the classical $UFLP$ model with a closest assignment constraint. Since $UFLP$ is NP-hard, $UFLPCOC$ is NP-hard as well. Thus, it is difficult to obtain good solutions for large size instances of $UFLPCOC$ within a limited time frame. This fact motivates research on approximate approaches.

The heuristic presented below is based on solutions to $LBMIP$ and its variant. Given Theorem 1, a solution of $LBMIP$ provides a lower bound for $UFLPCOC$. Note that the location and allocation decisions of this solution constitute a solution feasible to $UFLPCOC$, the objective value of which bounds the true optimal objective value of $UFLPCOC$ from above.

Denote by S the set of open facility locations under a location decision vector $\mathbf{x} = \{x_1, x_2, \cdots\}$. To find an improved upper bound, the heuristic proposed uses a descent search approach in the neighborhood of S. For each location set in the neighborhood we find its $UFLPCOC$ objective value. The neighborhood of a location set S and the descent approach are described as follows.

Define $N_r(S)$, the *distance-* r *neighborhood* of $S \subseteq M$ as

$$N_r(S) = \{S' \subseteq M: |S - S'| + |S' - S| \leq r\},$$

i.e. S' is in the distance- r neighborhood of S if the number of non-overlapping elements in the two sets does not exceed r.

Once the neighborhood is well defined, the descent search algorithm is straightforward: construct the neighborhood of set S; evaluate the objective value for every set in the neighborhood; move to the best set if the objective value improves. Repeat the process with the new set as the starting point until no improvement can be made. The best set obtained is the solution. We can then update the upper bound.

3.3 An Exact Solution Approach for $UFLPCOC$

The exact solution approach presented here is an iterative procedure that attempts to successively improve the lower and upper bounds for $UFLPCOC$ by applying the heuristic proposed in Section 3.2. In each step, we try to increase the lower bound by solving a revised $LBMIP$ model, which is a $LBMIP$ formulation with additional cuts to exclude pre-examined location vectors (at the first step the $LBMIP$ model does not include any cuts). The location set optimal to the revised $LBMIP$ model in a step that increases the lower bound then serves as the starting point to initialize the descent search process with an attempt to decrease the upper bound. The procedure continues until the lower bound is greater than the upper bound, so that it is evident that the unexamined location sets are unable to improve the current upper bound.

Next, we formulate the revised $LBMIP$ model in the l-th step of the algorithm, denoted by $LBMIP(l)$:

$$\min \sum_{j\in M} f_j x_j + \sum_{j\in M} V_j + \sum_{i\in N}\sum_{j\in M} \lambda_i c_{ij} y_{ij}$$

s.t.

$$(1), (2), (3), (4)$$

$$V_j + (1 - z_{jk})B \geq \textstyle\sum_{i\in N} \lambda_i y_{ij} p_{jk} + q_{jk}, \qquad \forall j \in M, 1 \leq k \leq K,$$

$$\textstyle\sum_{1\leq k\leq K} z_{jk} \geq x_j, \qquad \forall\, j \in M,$$

$$0 \leq z_{jk} \leq 1, \qquad \forall\, j \in M, 1 \leq k \leq K.$$

In the l-th step of the algorithm, an additional constraint set on the location decision variables, $X(l)$, is added to the revised $LBMIP$ model, denoted by $LBMIP(l)$. $X(l)$ contains a cut for each starting location set used in the descent search approach applied in steps 1 to $l-1$. Let $\Omega_{LBMIP(l)}$ denote all these locations sets. Recall the definition of the *distance-* r *neighborhood*. For any location set $S_{\hat{\mathbf{x}}} \in \Omega_{LBMIP(l)}$, the following constraint will ensure that any location decision vector $\mathbf{x}$ in the neighborhood of $\hat{\mathbf{x}}$ (and has already been examined) is infeasible:

$$\textstyle\sum_{j\in S_{\hat{\mathbf{x}}}} x_j - \sum_{j\in M-S_{\hat{\mathbf{x}}}} x_j \leq |S_{\hat{\mathbf{x}}}| - r - 1. \qquad (8)$$

It follows that $X(l)$ includes constraint (8) to exclude any location set $S_{\hat{\mathbf{x}}} \in \Omega_{LBMIP(l)}$. The addition of these cuts to the $LBMIP$ model will help to improve the lower bound.

Assuming a relative precision level η, we describe the exact solution approach below.

The Search and Cut Algorithm

- Step-0. **Let $l = 1$, $S_{\mathbf{x}^*} = \{\}$, $Lower\ Bound = 0$ and $Upper\ Bound = \infty$.**
- Step-1. **Solve $LBMIP(l)$. If the problem is feasible, denote the optimal objective value by $\underline{Z}^*_{LBMIP(l)}$ and optimal set of facility locations by $S_{\mathbf{x}^*_{LBMIP(l)}}$. Otherwise, go to Step 5.**
- Step-2. **If $ower\ Bound > \underline{Z}^*_{LBMIP(l)}$, then let $Lower\ Bound = \underline{Z}^*_{LBMIP(l)}$. If $Lower\ Bound > Upper\ Bound \times (1-\eta)$, then go to Step 5. Otherwise, go to step 3.**
- Step-3. **Apply the descent search approach on the neighborhood $N_r(S)$ starting from $S = S_{\mathbf{x}^*_{LBMIP(l)}}$. For every solution in $N_r(S)$, compute the objective value of $UFLPCOC$. Denote by $Z^*_{UFLPCOC}(l)$ the best objective value obtained. Construct set $\Omega_{LBMIP(l)}$.**
- Step-4. **If $Z^*_{UFLPCOC}(l) < Upper\ Bound$, then let $Upper\ Bound = Z^*_{UFLPCOC}(l)$ and update the incumbent solution accordingly. Let $l = l + 1$ and go to Step 1.**
- Step-5. **Stop. Return the incumbent solution and the upper bound as the optimal solution and the optimal objective value, respectively.**

The algorithm terminates when all potential location vectors have been examined or when the gap between the bounds is close enough. It is obvious that the algorithm terminates in a finite number of steps. Note that $Upper\ Bound$ represents the true objective value of the best-found feasible solution, while $Lower\ Bound$ is a valid lower bound on all unexamined location vectors.

IV. COMPUTATIONAL RESULTS

The Search and Cut algorithm has been coded in C++ and experiments have been conducted on a PC $Intel^{®}$ $Core^{TM}$ i7-2600K, 16GB of RAM. We use the optimization engine CPLEX v12.7 for solving all optimization problems.

The Search and Cut algorithm has been tested on four data sets that are based on 1990 census data, with each node representing one of the n largest cities in the United States. Demands λ_i are set to the city population divided by 10^4, and the fixed cost f_j is set to the median home value in the city. The transportation cost c_{ij} is calculated using the great circle distance between node i and j. In all four data sets, the set of facilities M is equal to the set of customers N.

We assume that the variable cost to operate facility j is given by the power function

$$F_j(\Lambda_j) = \beta\Lambda^{\alpha},\ \forall j \in M$$

with $\alpha = 0.5$ and $\beta = 2400$.

Three cost structures have been considered in the experiments for each data set with structure 1 being the baseline. In the other two structures, the fixed cost f_j and the variable cost $F_j()$ are amplified by ten times, respectively. So there are 12 problems in total. For reach problem, both approaches introduced in Section 3.2 have been employed to approximate variable cost $F_j()$ with the number of line segments $K_j \in \{1,3,5\}$. In the procedure to optimize the break points, we set $\varepsilon = 0.01$, $\xi = 2$ and the maximum number of iterations equal to 2000. For the Search and Cut algorithm with the neighborhood size $r \in \{1,2,3\}$, the relative precision level η has been chosen to be 1%.

In total, 180 instances were solved in the computational study. A clock time limit of 1800 seconds has been set for solving each MIP model as long as a feasible solution is returned. A clock time limit of 3600 seconds has been set for applying the Search and Cut algorithm. Note that the Search and Cut algorithm may not return a true optimal solution due to the time limit unless the gap between the upper bound and the lower bound is less than η or the upper bound is lower than the lower bound obtained. If the time limit was reached, we then searched for the best solution among the solutions returned by the Search and Cut algorithm and the approximation approaches (i.e., to solve the $LBMIP$ model with a clock time limit of 1800 seconds). Tables 2 to 5 present the clock time in seconds, the hit rate (hr, the percentage of instances where the best-known solution was returned) and the solution quality (err, the average relative error of the solution returned with respect to the best-known solution). The results yielded by the approximation solution approaches are also included. In the tables, the letter "n" or "y" following the number of line segments indicates, respectively, that the break points used in linear approximation are evenly spaced or optimized. The observations below can be made.

- Under cost structure 2, the fixed cost dominates the variable cost. As it would be suboptimal to open too many facilities, the problem might become easy to solve for a smaller solution space. As a result, all 60 instances were solved quickly to optimality by both solution approaches.
- The problem with a higher variable cost appears to be harder to solve than the baseline. In general, both solution approaches took longer clock time to solve the problem under cost structure 3 and the solution quality also deteriorated.

TABLE 2. SUMMARY OF COMPUTATIONAL RESULTS ($n = 50$)

Cost Structure	K_j	Search and Cut			*LBMIP*		
		time (s)	hr (%)	err (%)	time (s)	hr (%)	err (%)
1	1	3600	100	0	1	100	0
	3 n	3600	100	0	6	100	0
	3 y	51	100	0	43	100	0
	5 n	3600	100	0	74	100	0
	5 y	1906	100	0	1867	100	0
2	1	1	100	0	1	100	0
	3 n	1	100	0	1	100	0
	3 y	1	100	0	1	100	0
	5 n	2	100	0	1	100	0
	5 y	4	100	0	2	100	0
3	1	3600	0	0.2	1	0	0.7
	3 n	3600	0	0.4	1800	0	0.5
	3 y	3600	0	0.2	1800	0	0.2
	5 n	3600	0	0.1	1800	0	0.1
	5 y	2452	100	0	1800	100	0

TABLE 3. SUMMARY OF COMPUTATIONAL RESULTS ($n = 75$)

Cost Structure	K_j	Search and Cut			*LBMIP*		
		time (s)	hr (%)	err (%)	time (s)	hr (%)	err (%)
1	1	3600	0	0.0	1	0	0.8
	3 n	3600	0	0.0	49	0	0.3
	3 y	3051	100	0	1503	100	0
	5 n	3600	0	0.0	1800	0	0.3
	5 y	1918	100	0	1800	0	0.0
2	1	3	100	0	1	100	0
	3 n	3	100	0	1	100	0
	3 y	3	100	0	3	100	0
	5 n	5	100	0	5	100	0
	5 y	5	100	0	4	100	0
3	1	3600	67	0.0	1	0	8.9
	3 n	3600	33	0.2	1800	0	4.0
	3 y	3600	33	0.0	1800	0	0.0
	5 n	3600	33	0.0	1800	0	0.0
	5 y	1869	100	0	1800	0	0.0

TABLE 4. SUMMARY OF COMPUTATIONAL RESULTS ($n = 100$)

Cost Structure	K_j	Search and Cut			*LBMIP*		
		time (s)	hr (%)	err (%)	time (s)	hr (%)	err (%)
1	1	3600	0	0.0	1	0	0.5
	3 n	3600	0	0.0	153	0	0.2
	3 y	3600	0	0.0	1800	0	0.2
	5 n	3600	0	0.0	1800	0	0.2
	5 y	1925	100	0	1800	100	0
2	1	15	100	0	1	100	0
	3 n	6	100	0	2	100	0
	3 y	8	100	0	4	100	0
	5 n	14	100	0	11	100	0
	5 y	15	100	0	11	100	0
3	1	3600	0	0.5	1	0	7.7
	3 n	3600	0	2.5	1800	0	7.5
	3 y	3600	33	0.0	1800	0	0.0
	5 n	3600	0	0.1	1800	0	0.2
	5 y	3600	100	0	1800	0	0.7

TABLE 5. SUMMARY OF COMPUTATIONAL RESULTS ($n = 150$)

Cost Structure	K_j	Search and Cut			*LBMIP*		
		time (s)	hr (%)	err (%)	time (s)	hr (%)	err (%)
1	1	3600	0	0.0	1	0	0.7
	3 n	3600	0	0.0	461	0	0.1
	3 y	3600	0	0.0	1800	0	0.0
	5 n	3600	100	0	1800	100	0
	5 y	3600	100	0	1800	100	0
2	1	56	100	0	1	0	0.0
	3 n	31	100	0	8	0	0.0
	3 y	49	100	0	9	100	0
	5 n	76	100	0	55	100	0
	5 y	88	100	0	29	100	0
3	1	3600	0	1.5	1	0	14.4
	3 n	3600	33	0.8	1800	0	1.3
	3 y	3600	66	0.4	1800	0	1.3
	5 n	3600	33	0.2	1800	0	3.0
	5 y	3600	100	0	1800	0	0.5

- The computational results not reported here showed that the number of facilities to open in the optimal solution or the best-known solution under cost structure 2 and structure 3 was usually 40 to 50 percent of that under the baseline. We could infer that fewer facilities were open so as to avoid high fixed costs or to take advantage of economies of scale with respect to variable costs.
- In general, the solution gap between the upper bound and the lower bound of the Search and Cut algorithm decreased with the neighborhood radius r. This seemed to suggest that descent search was effective to improve the upper bound.
- In general, the solution gap of the Search and Cut algorithm between the upper bound and lower bound decreased with the number of line segments K_j. Optimizing the break points also had the potential to reduce the gap.
- The best objective value of the $LBMIP$ model returned in general improved (i.e., getting smaller) as the number of line segments K_j increased and after the break points were optimized. However, this trend discontinued at $n = 150$ possibly due to the time limit set on the Branch and Bound algorithm.
- For both two solution approaches, optimizing the break points in linear approximation helped improve the solution quality.
- Compared to the approximation solution approach, the Search and Cut algorithm returned either the same solution or a better solution for all instances.
- For all 12 problems, the Search and Cut algorithm with $K_j = 5$ line segments and optimal break points returned the optimal solution or the best-known solution under all three neighborhood size levels.

Based on the above observations, we would like to recommend the Search and Cut algorithm for solving the $UFLPCOC$ model after approximating each variable cost function with $K_j = 5$ line segments and optimizing the break points.

V. CONCLUSIONS

An uncapacitated facility location model with concave operating costs is presented. We require that customers travel to the closest open facility for service. Therefore, a constraint is incorporated into the optimization model to impose the closest assignment property. An exact approach for solving the $UFLPCOC$ model is developed. The approach is based on obtaining efficient lower and upper bounds. Lower bounds are found by solving an MIP model with a tighten feasible region in each step of the algorithm. This is achieved by introducing cutting planes in the MIP model. On the other hand, a heuristic based on neighborhood search is proposed to find an upper bound.

Computational experiments suggest that the cost structure of the problem strongly influences the difficulty in solving $UFLPCOC$ exactly. We observe that solving the problem to optimality is harder when variable costs dominate fix costs. It is evident that when approximating each variable cost function using five line segments and optimizing the break points, the Search and Cut algorithm is able to return a high quality solution.

As an extension of this work, we plan to apply the methodology proposed to other types of functions, such as difference of convex (d.c.) functions (Blanquero and Carrizosa, 2009). This functions can be easily approximated by a linear function in the same way we have done with concave functions. As another possible extension, we would like to apply our solution method to other problems of the same nature, such as p-median type problems with nonlinear costs, as in Carrizosa, Ushakov and Vasilyey (2012), or a different nature. We can thus compare the performance of our approach and other available methods.

REFERENCES

Aboolian, R., Cui, T. and Shen, Z.M., "An Efficient Approach for Solving Reliable Facility Location Models", *INFORMS Journal on Computing*, 25, 2012, 720-729.

Berman, O. and Krass, D., Stochastic Location Models with Congestion, In Laporte G. et al. (ed.), *Location Science*, Springer International Publishing, Switzerland, 2015, 443-486.

Berman, O., Krass, D. and Wang, J., "Locating Service Facilities to Reduce Lost Demand", *IIE Transactions*, 38, 2006, 933-946.

Blanquero, R. and Carrizosa, E., "Continuous Location Problems and Big Triangle Small Triangle: Constructing Better Bounds", *Journal of Global Optimization*, 45, 2009, 389-402.

Brandeau, M. and Chiu, S., "An Overview of Representative Problems in Location Research", *Management Science*, 35, 1989, 645-674.

Carrizosa, E., Ushakov, A. and Vasilyev, I., "A Computational Study of a Nonlinear Minsum Facility Location Problem", *Computers and Operations Research*, 39, 2012, 2625-2633.

Cornuejols, G., Fisher, M.L. and Nemhauser, G.L., "Location of Bank Accounts to Optimize Float: An Analytic Study of Exact and Approximate Algorithms", *Management Science*, 23, 1977, 789-910.

Efroymson, M.A. and Ray, T.L., "A Branch-and-bound Algorithm for Plant Location", *Operations Research*, 14, 1966, 361-368.

Khumawala, B.M. and Kelly, D.L., "Warehouse Location with Concave Costs", *INFOR*, 12, 1974, 55-65.

Krarup, J. and Pruzan, P., "The Simple Plant Location Problem: Survey and Synthesis", *European Journal of Operational Research*, 12, 1983, 36-81.

Morris, J.G., "On the Extent to Which Certain Fixed-charge Depot Location Problems Can Be Solved by LP", *Journal of the Operational Research Society*, 29, 1978, 71-76.

Soland, R.F., "Optimal Facility Location with Concave Costs", *Operations Research*, 22, 1974, 373-385.

Verter, V., Uncapacitate and Capacitated Facility Location Problems, In Eiselt, H.A. and Marianov, V. (ed.), *Foundations of Location Analysis*, Springer Science+Business Media, New York, 2001, 25-37.

Verter, V. and Dincer M.C., "Facility Location and Capacity Acquisition: An Integrated Approach", *Naval Research Logistics*, 42, 1995, 1141-1160.

William, H.P., "Experiments in the Formulation of Integer Programming Problems", *Mathematical Programming Study*, 2, 1974, 180-197.