A Simple EOQ-like Solution to an Inventory System with Compound Poisson and Deterministic Demand

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We consider a continuous review inventory system where demand is a mixture of a deterministic component and a random component that follows a compound Poisson process. Unmet demand is backordered and a two-number (*s*,*S*) policy is used to control inventory. When inventory drops below level *s*, an order is placed to replenish up to level *S*. The steady state distribution of inventory level is derived using the level crossing approach. We establish that this distribution is an increasing exponential function of inventory level. From the steady state distribution, the exact total expected cost function, consisting of setup costs and inventory holding and backorder costs, is determined. We propose a heuristic that gives a simple closed-form solution with near-optimal performance. We find that the standard EOQ with backorders overestimates *S* and underestimates *s* and can be a poor approximation to the optimal policy.

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I. INTRODUCTION

The problem we study is a continuous review inventory system for a single product with a fixed setup cost for ordering, and linear inventory holding and backorder costs. Demand for the product is a mixture of two parts: a constant deterministic component and a random component. We assume that the random component follows a compound Poisson process and the size of each demand request is an exponential random variable. This type of demand process is seen in practice where a firm sells a product through two different channels. One channel generates more or less constant demand from long-standing contracts, while the other channel represents customers with unscheduled random orders. Sobel and Zhang (2001) provide additional examples.

We note that the addition of a constant demand component to the compound Poisson demand might seem innocuous at first glance; however, it complicates the analysis because system transitions can occur continuously in time in addition to those that occur at demand arrival epochs. See Presman and Sethi (2006) for further support of this point.

We assume that the inventory control policy is of the (s, S) type. When inventory goes below level s, an order is placed to bring the inventory level up to level S. We assume that deliveries are instantaneous. Under our demand and cost structure, namely, mixture of constant and Poisson demand components, fixed ordering, linear inventory holding, and backorder costs, Presman and Sethi (2006) show that an (s, S) policy minimizes the long run average costs.

The standard EOQ model with backorders, which assumes that demand is both deterministic and uniform, is often used, even when demand is uncertain, by using the expected demand rate in place of the deterministic demand rate. This approach may not work well when demand has a high degree of randomness. As we shall see, the EOQ solution overestimates S and underestimates s and leads to poor performance when demand exhibits relatively large random jumps.

In this paper, we derive an exact expression for the stationary distribution of inventory level. We also develop a near-optimal simple, closed-form solution to our problem. This solution takes into account the effect of randomness in the demand process and offers near-optimal performance without any additional computational burden over the standard EOQ model with backorders.

As noted before, our demand structure, where transitions can occur at demand arrival epochs (i.e. Poisson arrivals) as well as continuously in time due to constant demand, adds complexity to the analysis. The continuous transitions make the process a continuous-time, continuous state space process and our problem cannot be modeled as a semi-Markov decision process. We find that the level crossing approach is useful to analyze our problem and derive the stationary distribution of demand. We describe this method later.

The paper is organized as follows. Section 2 reviews the related literature. In Section 3, we present the inventory problem and use the level crossing approach to derive the stationary distribution of inventory level. In Section 4, the exact expected cost function is presented, from which the optimal (s, S) policy is determined. In Section 5, we describe our heuristic and derive a simple closed-form solution for the (s, S) policy with near-optimal performance. Section 6 has a full-factorial numerical study that shows the effectiveness of our heuristic policy. In Section 7, we consider some special cases. We show how our analysis applies to the case where demand is only a compound Poisson process (no constant component) and we give a simple closed-form solution for the optimal (s, S) policy. We also show how our model reduces to the EOQ model when demand is only deterministic. Finally, in Section 8, we present concluding remarks.

II. LITERATURE REVIEW

The literature on continuous review inventory systems is vast. Here we limit the discussion to relevant areas of research.

In the following, we discuss work that models demand as mixture of constant and compound Poisson processes. Presman and Sethi (2006) prove that an (s, S) policy is optimal in a continuous review inventory system with a fixed cost of ordering. Sobel and Zhang (2001) study a periodic review inventory model and prove that a modified (s, S) policy is optimal if there is a fixed cost of ordering. The above work establishes the optimality of the (s, S) policy with this demand process, but does not provide the stationary distribution of inventory level, nor closed-form solutions for the policy. Our analysis in this paper derives, for the first time, exact expressions for the stationary distribution of inventory and expected costs. We find that the stationary distribution is an exponentially increasing function of inventory level. We propose an approximate expected cost function that allows for a simple closed-form solution that looks like the EOQ with backorders.

To derive the stationary distribution of inventory, we use the level crossing method, which was initially developed to obtain the stationary probability distribution of waiting time in queues (Brill and Posner 1977, 1981). This method analyzes a stochastic process where the state of the process undergoes upward or downward jumps according to a Poisson process. Based on the sample path, one develops balance equations from which the steady-state distribution of waiting times can be derived. In addition to applications in queuing, this method has been used in other areas such as inventory systems, risk models in insurance, and control policies for a dam. An overview of this method and applications is in Brill (2008).

As for applications of the level crossing method in inventory problems, we discuss a few representative papers here. The first such work by Azoury and Brill (1986) derives the steady-state distribution of inventory level for a perishable product. Other work in this area include Azoury and Brill (1992) who study an inventory system with random lead time, Brill and Chaouch (1995) who consider variations in demand rate at random points in time, and Mohebbi and Posner (1999) who study lost sales and emergency orders. Chaouch (2001) studies an inventory system where demand is modeled as a mixture of constant and compound Poisson processes, deliveries from the supplier to the retailer follow a Poisson process at rate θ, and, at each replenishment epoch, inventory is brought up to a fixed level S. This replenishment policy is not of the (s, S) type. Instead, the decision variables are the rate of delivery θ and the fixed order-up-to level S. More recently, Chaouch (2007) analyzes an inventory model where sellers offer price-discounting strategies at random times to clear inventory.

The work in this paper is an extension of Azoury et al (2012). In Azoury et al (2012) demand was also modeled as a mixture of constant and compound Poisson processes, but backorders were not allowed (i.e. $s = 0$). When backorder costs are finite, the assumption of not allowing backorders can lead to large errors (Gallego and Roundy, 1992). Here we explicitly incorporate backorders into the model

III. STEADY-STATE ANALYSIS OF INVENTORY PROBLEM

3.1 Description of Inventory Problem

The problem that we study is a continuous review inventory system for a single product. Demand for the product is a mixture of two demand components: a deterministic component with a constant demand rate D and a random component where demand arrives according to an exponential distribution with rate λ and each demand size is exponential with mean $\frac{1}{\mu}$. There is fixed cost K each time an order is placed and we assume instantaneous replenishment (i.e. no lead-time). Demand is backordered when there is not sufficient inventory on hand. We assume linear inventory holding and backorder costs per unit time, denoted by h and b respectively.

The state space for this inventory system is characterized by the inventory level x. We assume that a two-critical number inventory policy (s, S) is used where an order is placed when the inventory level goes below level s and enough is ordered to bring the inventory up to level S.

We note that the optimal value for s will always be less than or equal to zero because an $s = 0$ will always outperform an s greater than zero. Neither $s = 0$ nor $s > 0$ results in backorder costs and $s = 0$ has lower inventory holding costs than $s > 0$.

In the following, we describe how the inventory level evolves over time within an order cycle. At the start of a cycle, the inventory level is S. Constant demand decreases the inventory level continuously and random demand drops the inventory in jumps at demand arrival epochs. When the inventory level falls below s, a replenishment order is triggered to bring the inventory level up to S. The sample path for the inventory level is in Figure 1, where x represents an inventory level between s and S.

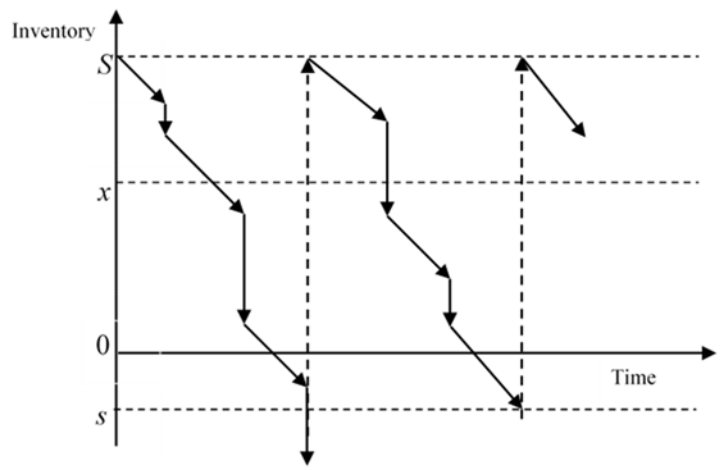

FIGURE 1. SAMPLE PATH FOR THE INVENTORY LEVEL.

3.2. Derivation of Steady-State Distribution of Inventory Level

Since demand has a random component, the inventory level $x(t)$ at time t is a stochastic process. We are interested in the steady-state distribution of the inventory level as $t \rightarrow \infty$. We note that a steady-state distribution exists because the inventory level will hit level S, which leads to regenerative cycles that start and stop at this state. To get this distribution, we apply the level crossing methodology to write balance equations by equating exit rates to entrance rates from inventory states. These balance equations are solved to obtain the stationary distribution of the inventory level. We use the following notation:

s = Reorder point level
S = Order-up-to level
h = Inventory holding cost per unit per unit time
b = Backorder cost per unit per unit time
K = Fixed cost per order
D = Demand rate for constant component of demand
λ = Poisson demand arrival rate
$\frac{1}{\mu}$ = Mean demand size of Poisson arrivals
$g(x)$ = Stationary distribution of the inventory level x

Referring to Figure 1, we equate the down-crossing and up-crossing rates for inventory level x between s and S. The down-crossing rate of level x is

$$Dg(x) + \lambda \int_x^S e^{-\mu(y-x)} g(y)dy$$
for $s \leq x < S$. (1)

The first term is the down-crossing rate of level x due to the continuous transitions in inventory level from the constant demand. The second term is the down-crossing rate of level x due to the jumps from the compound

Poisson demand. We note that the integral in the second term represents the probability of being at any inventory level y between x and S and having a demand jump bigger than $y - x$. The up-crossing rate of level x equals the down-crossing rate of level s, which is

$$Dg(s) + \lambda \int_s^S e^{-\mu(y-s)} g(y)dy. \quad (2)$$

Equating down-crossing and up-crossing rates of level x gives:

$$Dg(x) + \lambda \int_x^S e^{-\mu(y-x)} g(y)dy = Dg(s) + \lambda \int_s^S e^{-\mu(y-s)} g(y)dy \quad \text{for } s \leq x < S. \quad (3)$$

We have another balance equation at level S, which we write as

$$Dg(S) = Dg(s) + \lambda \int_s^S e^{-\mu(y-s)} g(y)dy. \quad (4)$$

The left-hand side of (4) represents the rate out of level S. This rate is due to only the continuous transitions out of level S from constant demand (no jumps). The right-hand side of (4) is the rate into level S, which is equal to the down-crossing rate of level s in (2). Note that if we plug in $x = S$ in (3), the down-crossing rate due to jumps (second term in left-hand side of (3)) is zero and we get (4); hence (3) is valid for any x where $s \leq x \leq S$.

Finally, the stationary distribution of the inventory level must satisfy the following normalizing condition:

$$\int_s^S g(y)dy = 1. \quad (5)$$

Equations (3), (4), and (5) are sufficient to determine the stationary distribution $g(x)$.

In the following, we show the steps that solve for $g(x)$. We convert the integral equation (3) into a differential equation by taking derivatives of both sides.

$$Dg'(x) + \lambda\mu \int_x^S e^{-\mu(y-x)} g(y)dy - \lambda g(x) = 0 \quad \text{for } s \leq x \leq S. \quad (6)$$

Using the balance equations in (3) and (4), we can rewrite (6) as

$$g'(x) = \left(\mu + \frac{\lambda}{D}\right) g(x) - \mu g(S) \quad \text{for } s \leq x \leq S. \quad (7)$$

To simplify notation we let R be defined in terms of the original demand parameters:

$$R = \mu + \frac{\lambda}{D}. \quad (8)$$

Equation (7) is a first order differential equation and using the standard solution approach results in

$$g(x) = Ae^{Rx} + \frac{1}{R}\mu g(S) \quad \text{for } s \leq x \leq S \quad (9)$$

where A is a constant. By substituting $x = S$ in (9), we can solve for the constant A.

$$A = \frac{\lambda}{DR} g(S) e^{-RS}. \quad (10)$$

Therefore,

$$g(x) = g(S)\left[\frac{\lambda}{DR} e^{-R(S-x)} + \frac{\mu}{R}\right] \quad \text{for } s \leq x \leq S. \quad (11)$$

Using (11) and the normalizing condition in (5), we can solve for $g(S)$.

$$g(S) = \frac{R}{\mu(S-s) + \frac{\lambda}{DR}\left(1 - e^{-R(S-s)}\right)}. \quad (12)$$

The density $g(x)$ is an increasing exponential function in x for $s \leq x \leq S$. In Fig. 2, we plot the density $g(x)$ for an example with the following parameters: $\lambda = 5$, $\mu = 0.01$, $D = 100$, $S = 200$, and $s = -10$.

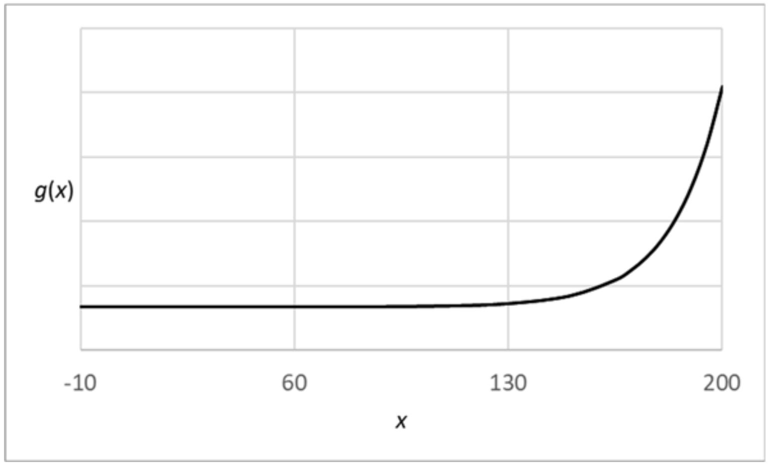

FIGURE 2. STATIONARY DISTRIBUTION OF INVENTORY LEVEL.

IV. EXPECTED COST FUNCTIONS

The expected cost function is the sum of the expected ordering, inventory holding, and backorder costs per unit time. We use the stationary distribution of inventory level, $g(x)$ in (11), to derive the expected cost functions in terms of s and S, which we denote by $Z(s,S)$. As noted earlier in Section 3.1, the optimal value for s will always be less than or equal to zero because an $s = 0$ will always outperform an s greater than zero. Thus, we do not need to consider values of s greater than zero. The expected cost function $Z(s,S)$ is

$$Z(s,S) = KDg(S) + h\int_0^S yg(y)dy - b\int_s^0 yg(y)dy. \qquad (13)$$

After substituting (11) in (13), computing the integrals, and simplifying, we can write

$$Z(s,S) = KDg(S) + hg(S)\left\{\frac{\mu}{2R}S^2 + \frac{\lambda}{DR^2}S - \frac{\lambda}{DR^3}\right\} + bg(S)\left\{\frac{\mu}{2R}s^2\right\} + bg(S)\left\{\frac{\lambda}{DR^2}se^{-R(S-s)} - \frac{\lambda}{DR^3}e^{-R(S-s)}\right\} + (h+b)g(S)\frac{\lambda}{DR^3}e^{-RS} \qquad (14)$$

where $g(S) = \frac{R}{\mu(S-s)+\frac{\lambda}{DR}(1-e^{-R(S-s)})}$ as shown in (12).

The optimal (s,S) policy minimizes $Z(s,S)$ in (14). Unfortunately, it is not possible to derive a closed-form solution for the optimal policy. Due to the form of $Z(s,S)$, it is not easy to show that the function is convex, but our computational experience indicates that there is a unique solution. Standard nonlinear search techniques can be used to find the optimal (s,S) policy and we outline an approach in the Appendix.

In the next section, we develop a convex approximation to $Z(s,S)$ in (14), from which near-optimal closed-form solutions are derived.

V. APPROXIMATION AND CLOSED-FORM SOLUTION

We develop an approximation to $Z(s,S)$ in (14) by noting that the exponential terms are negligible. First, we make a modest assumption that, for typical applications, S is reasonably large such that e^{-RS} is negligible. (Note that R is positive.) This assumption is backed by extensive numerical studies on a wide range of parameter values. Since $s \leq 0$, then it follows that $e^{-R(S-s)}$ is also negligible. From this assumption, it follows that the approximate expected cost function, which we denote by $\hat{Z}(s,S)$, can be written as

$$\hat{Z}(\mathrm{s},\mathrm{S}) = KD\hat{g}(S) + h\hat{g}(S)\left\{\frac{\mu}{2R}S^2 + \frac{\lambda}{DR^2}S - \frac{\lambda}{DR^3}\right\} + b\hat{g}(S)\left\{\frac{\mu}{2R}s^2\right\} \quad (15)$$

where $$\hat{g}(S) = \frac{R}{\mu(S-s)+\frac{\lambda}{DR}}. \quad (16)$$

Here $\hat{g}(S)$ is an approximation to $g(S)$ after dropping the exponential term.

To establish convexity of the approximate cost function and to derive the closed-form solutions, we find that an alternative representation is more convenient. This alternative representation involves the following change of variable, defining Q as follows:

$$Q = \frac{\lambda}{\mu DR} + S - s. \quad (17)$$

Applying the change of variable, Q in (17), to (15) results in

$$\hat{Z}(s,Q) = \frac{1}{Q}\left\{K\left(D+\frac{\lambda}{\mu}\right) - h\left(\frac{\lambda^2}{2\mu^2D^2R^2} + \frac{\lambda}{\mu DR^2}\right) + (h+b)\frac{s^2}{2}\right\} + \frac{h}{2}Q + hs. \quad (18)$$

Note that, for a given s, the expected cost function in (18) has the same form as the classic EOQ model with a hyperbolic term in Q and a linear term in Q. With the representation of $\hat{Z}(s,Q)$ in (18), Q is the expected order quantity.

The function $\hat{Z}(s,Q)$ in (18) is convex in s and Q under the following mild condition:

$$\frac{2K\left(D+\frac{\lambda}{\mu}\right)}{h} - \frac{\lambda^2}{\mu^2D^2R^2} - \frac{2\lambda}{\mu DR^2} > 0. \quad (19)$$

See Appendix for details.

From the first order conditions of (18), we get the following closed-form solution for minimizing (18).

$$\hat{Q} = \sqrt{\left(\frac{h+b}{b}\right)\left(\frac{2K\left(D+\frac{\lambda}{\mu}\right)}{h} - \frac{\lambda^2}{\mu^2D^2R^2} - \frac{2\lambda}{\mu DR^2}\right)},$$

$$\hat{s} = -\frac{h}{h+b}\hat{Q}, \text{ and } \hat{S} = \hat{Q} + \hat{s} - \frac{\lambda}{\mu DR}. \quad (20)$$

The convexity condition in (19), assures that (20) is a unique solution to (18). In the next section, we perform a numerical study and show near-optimal performance of this closed-form solution.

VI. NUMERICAL STUDY

We designed a full factorial numerical study by varying the parameters K, b, λ, μ, and D, each over two settings, one low and one high. Without loss of generality, we fixed h at 1. The reason is that, of the three cost parameters, we only have to vary two of them. The parameter settings are shown in Table 1.

TABLE 1. NUMERICAL STUDY PARAMETER SETTINGS.

	K	h	b	λ	μ	D
Low	200	1	5	1	0.01	5
High	1000	1	100	10	0.10	20

A full factorial of the parameters settings in Table 1 generated 32 trials.

The purpose of this study is to assess how well our heuristic policy performs compared to the optimal policy. First, we determined the optimal policy (via search) that minimizes the exact expected cost function $Z(s,S)$ in (14). Then, we calculated the heuristic policy from the closed-form solution in (20). For each of the two policies, we computed the exact expected total cost and calculated the error from optimal for the heuristic policy, namely $\frac{Z(\hat{s},\hat{S})-Z(s^*,S^*)}{Z(s^*,S^*)}$. In Table 2 we summarize the results. The optimal policy, the heuristic policy and the percentage error of the heuristic are shown in the last three columns.

TABLE 2. NUMERICAL STUDY RESULTS.

Trial #	Model parameters					Optimal (s^*, S^*)	Heuristic $(\hat{s}, \hat{S})$	Heuristic Error (%)
	K	b	D	λ	μ			
1	200	5	5	1	0.01	(-33, 68)	(-33, 68)	0.000%
2	200	5	5	1	0.10	(-14, 64)	(-14, 64)	0.000%
3	200	5	5	10	0.01	(-114, 472)	(-114, 472)	0.000%
4	200	5	5	10	0.10	(-37, 177)	(-37, 177)	0.000%
5	200	5	25	1	0.01	(-37, 106)	(-37, 103)	0.007%
6	200	5	25	1	0.10	(-22, 105)	(-22, 105)	0.000%
7	200	5	25	10	0.01	(-115, 480)	(-115, 480)	0.000%
8	200	5	25	10	0.10	(-41, 196)	(-41, 196)	0.000%
9	200	100	5	1	0.01	(-2, 83)	(-2, 83)	0.000%
10	200	100	5	1	0.10	(-1, 70)	(-1, 70)	0.000%
11	200	100	5	10	0.01	(-6, 523)	(-6, 523)	0.000%
12	200	100	5	10	0.10	(-2, 194)	(-2, 194)	0.000%
13	200	100	25	1	0.01	(-2, 122)	(-2, 120)	0.006%
14	200	100	25	1	0.10	(-1, 115)	(-1, 115)	0.000%
15	200	100	25	10	0.01	(-6, 532)	(-6, 532)	0.000%
16	200	100	25	10	0.10	(-2, 214)	(-2, 214)	0.000%
17	1000	5	5	1	0.01	(-82, 313)	(-82, 313)	0.000%
18	1000	5	5	1	0.10	(-32, 151)	(-32, 151)	0.000%
19	1000	5	5	10	0.01	(-258, 1191)	(-258, 1191)	0.000%
20	1000	5	5	10	0.10	(-84, 409)	(-84, 409)	0.000%
21	1000	5	25	1	0.01	(-90, 368)	(-90, 368)	0.000%
22	1000	5	25	1	0.10	(-48, 239)	(-48, 239)	0.000%
23	1000	5	25	10	0.01	(-261, 1206)	(-261, 1206)	0.000%
24	1000	5	25	10	0.10	(-91, 448)	(-91, 448)	0.000%
25	1000	100	5	1	0.01	(-4, 350)	(-4, 350)	0.000%
26	1000	100	5	1	0.10	(-2, 165)	(-2, 165)	0.000%
27	1000	100	5	10	0.01	(-14, 1308)	(-14, 1308)	0.000%
28	1000	100	5	10	0.10	(-5, 446)	(-5, 446)	0.000%
29	1000	100	25	1	0.01	(-5, 408)	(-5, 408)	0.000%
30	1000	100	25	1	0.10	(-3, 260)	(-3, 260)	0.000%
31	1000	100	25	10	0.01	(-14, 1324)	(-14, 1324)	0.000%
32	1000	100	25	10	0.10	(-5, 489)	(-5, 489)	0.000%

Our heuristic policy is essentially the same as the optimal policy. In 30 of the 32 trials, the heuristic is identical to the optimal policy. There is a minor difference between the heuristic and the optimal in trials 5 and 13. The largest error was 0.007% in Trial 5.

As mentioned earlier, the standard EOQ policy is often used because of its simplicity. The EOQ model with backorders has the following cost function and optimal policy:

$$Z_E(s,Q) = \frac{1}{Q}\left\{K\Omega + (h+b)\frac{s^2}{2}\right\} + \frac{h}{2}Q + hs \qquad (21)$$

$$Q_E = \sqrt{\left(\frac{h+b}{b}\right)\left(\frac{2K\Omega}{h}\right)},$$
$$s_E = -\frac{h}{h+b}Q_E, \text{ and } S_E = Q_E + s_E \qquad (22)$$

where Ω represents the demand rate and the subscript E refers to the EOQ. If we were to use the policy in (22) in our setting, we would set Ω equal to our expected demand, $D + \frac{\lambda}{\mu}$. We tested this EOQ policy in all 32 trials and the performance was suboptimal with errors as high as 13.7%.

Comparing the closed-form solutions in (20) and (22), we see that the order quantity from the EOQ is larger than the expected order quantity in our heuristic (i.e. $Q_E > \hat{Q}$). It is also clear from (20) and (22) that $s_E < \hat{s}$, and $S_E > \hat{S}$. Hence, the EOQ simplification is biased on the side of lower reorder levels and higher order-up-to levels as compared to our heuristic. Our heuristic has a simple closed-form solution like the EOQ, accounts for randomness in demand, and provides near-optimal performance. Hence, our heuristic offers the simplicity of the EOQ with the added benefit of near-optimal performance.

VII. SPECIAL CASES INCLUDING THE EOQ MODEL

Case when D = 0 (only compound Poisson demand): For the case of only compound Poisson demand, we derive a closed-form solution that is optimal. In this case, there is a nonzero probability that the inventory level is at S because when the inventory level hits S it remains there until a demand jump occurs. We let Π denote the probability that inventory is at level S. Using the level crossing approach, it can be shown that the stationary distribution of inventory level has a point mass at level S and is uniform for inventory levels x, $s \leq x < S$. The details are in the Appendix. The stationary distribution for level x is

$$\Pi = \frac{1}{1+\mu(S-s)} \quad \text{for } x = S$$
$$\text{and } g(x) = \frac{\mu}{1+\mu(S-s)} \text{ for } s \leq x < S. \qquad (23)$$

We note that this density function cannot be obtained by plugging D = 0 in $g(x)$ in (11) because of the point mass at level S. Hence, the case of D = 0 requires its own balance equations.

The expected cost function $Z(s,S)$ is

$$Z(s,S) = \Pi\left[K\lambda + \frac{h}{2}\mu S^2 + hS + \frac{b}{2}\mu s^2\right]. \qquad (24)$$

Applying the change in variable, $Q = \frac{1}{\mu} + S - s$, to (24) results in

$$Z(s,Q) = \frac{1}{Q}\left\{K\frac{\lambda}{\mu} - \frac{h}{2\mu^2} + (h+b)\frac{s^2}{2}\right\} + \frac{h}{2}Q + hs. \qquad (25)$$

It is straightforward to show that $Z(s,Q)$ is a convex function in s and Q. From the first order conditions the optimal solution is

$$Q^* = \sqrt{\left(\frac{h+b}{b}\right)\left(\frac{2K\left(\frac{\lambda}{\mu}\right)}{h} - \frac{1}{\mu^2}\right)},$$

$$s^* = -\frac{h}{h+b}Q^*, \text{ and } S^* = Q^* + s^* - \frac{1}{\mu}. \quad (26)$$

Case when λ = 0 (only deterministic demand): For the case of only deterministic demand, by setting $\lambda = 0$ in (11), $g(x)$ reduces to $\frac{1}{(S-s)}$ for $s \le x < S$. The cost function in (14) reduces to the EOQ with backorders cost function in (21) by setting $\lambda = 0$ in (14) and applying the change in variable, $Q = S - s$.

VIII. CONCLUSION

In this paper, we studied a continuous-review inventory system where demand is a mixture of a deterministic component and a compound Poisson stochastic component. Unmet demand is backordered. We assumed a two-critical number policy (s, S). If inventory is below level s, then an order is placed to bring the inventory up to level S. We used the level crossing method to derive the steady-state distribution of the inventory level. We then derived the exact expression for the expected cost as a function of s and S. We outlined how the optimal (s, S) policy can be determined from the expected cost function. In addition, we proposed an approximate expression for the expected cost function that is easy to minimize with a near-optimal, closed-form solution for (s, S). We ran a numerical study to verify the near-optimal performance of our closed-form formulas. In the case of only Poisson compound demand, we present closed-form solutions for the optimal policy. Finally, we demonstrated that using the standard EOQ formula with backorders might be quite suboptimal.

One direction for future research is to extend the work in this paper to the case of perishable inventory. Another direction is to generalize the demand process to multiple compound Poisson demands in addition to the deterministic component. As an example you could have two different types of customers each with its own Poisson arrivals (λ_1 and λ_2) and demand jumps (μ_1 and μ_2). In both of the these extensions, we believe that the level crossing approach provides a promising tool to derive the steady state distribution of inventory level from which performance measures such as expected costs can be determined.

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APPENDIX

Optimality of $Z(s,S)$:

To outline our steps for optimality, we find that an alternative representation of $Z(s,S)$ in (14) is more convenient. This alternative representation involves the following change of variable, defining Q as follows:

$$Q=\frac{\lambda}{\mu DR}+S-s. \tag{A1}$$

With this change of variable, we write (12) in terms of Q.

$$g(S)=\frac{R}{\mu\left(Q-\frac{\lambda}{\mu DR}e^{-R(Q-C)}\right)} \tag{A2}$$

where $C=\frac{\lambda}{\mu DR}$. Note that C represents the minimum value of Q as defined in (A1) because $S-s\geq 0$.

Next, we write $Z(s,S)$ as a function of s and Q, that is, $Z(s,Q)$.

$$\begin{aligned}Z(s,Q)=&\left(\frac{1}{Q-\frac{\lambda}{\mu DR}e^{-R(Q-C)}}\right)\Big\{K\left(D+\frac{\lambda}{\mu}\right)\\&+h\left(\frac{Q^2}{2}+sQ-\frac{\lambda^2}{2\mu^2D^2R^2}-\frac{\lambda}{\mu DR^2}\right)\\&+(h+b)\frac{s^2}{2}\\&+b\left(\frac{\lambda}{\mu DR}se^{-R(Q-C)}-\frac{\lambda}{\mu DR^2}e^{-R(Q-C)}\right)\\&+(h+b)\frac{\lambda}{\mu DR^2}e^{-R(Q+s-C)}\Big\}.\end{aligned} \tag{A3}$$

In the following, we explain how to find the optimal (s,Q) policy which we denote by (s^*, Q^*). It can be shown that for a given Q, $Z(s,Q)$ in (A3) is convex with respect to s. The optimal value of s satisfies s^* = min(0, s) where s meets the first order condition of $Z(s,Q)$ with respect to s.

$$\begin{aligned}&\frac{\lambda}{\mu DR}e^{-R(Q+s-C)}-s\\&=\frac{h}{h+b}Q+\frac{b}{(h+b)}\frac{\lambda}{\mu DR}e^{-R(Q-C)}.\end{aligned} \tag{A4}$$

Using (A4), the expected cost function in (A3) can be searched over Q to find the optimal (s,Q) policy. The optimal value of S, which we denote by S^*, follows from the change in variable in (A1).

Convexity of $\hat{Z}(s,S)$

$\hat{Z}(s,Q)$ in (18) is convex if its Hessian is positive definite. The Hessian is positive definite if $\frac{\partial^2\hat{Z}(s,S)}{\partial s^2}\geq 0$ and the determinant of the Hessian is greater than or equal to zero. It can be shown that

$$\frac{\partial^2 \hat{Z}(s,S)}{\partial s^2} = \frac{1}{Q}(h+b) > 0$$

and the determinant of the Hessian matrix of $\hat{Z}(s,Q)$ is

$$\frac{1}{Q^4}(h+b)h\left[\frac{2K\left(\frac{\lambda}{\mu}+D\right)}{h} - \frac{\lambda^2}{\mu^2 D^2 R^2} - \frac{2\lambda}{\mu D R^2}\right]$$

which is greater than or equal to zero if the expression in brackets is greater than or equal to zero.

Derivation of the steady-state distribution of the inventory level for the compound Poisson demand case.

Equating down-crossing and up-crossing rates of level x gives

$$\lambda \Pi e^{-\mu(S-x)} + \lambda \int_x^S e^{-\mu(y-x)} g(y)dy$$
$$= \lambda \Pi e^{-\mu(S-s)} + \lambda \int_s^S e^{-\mu(y-s)} g(y)dy$$
for $s \le x < S$. (A5)

Equating the rate out of level S to the rate into level S results in

$$\lambda \Pi = \lambda \Pi e^{-\mu(S-s)} + \lambda \int_s^S e^{-\mu(y-s)} g(y)dy. \quad \text{(A6)}$$

We note that the right-hand side of (A5) is the down-crossing rate of level s, which is equal to the up-crossing rate of any level x, for $s \le x < S$. Similarly, the right-hand side of (A6) is equal to the rate into level S. The distribution of the inventory level must satisfy the normalizing condition:

$$\Pi + \int_s^S g(y)dy = 1. \quad \text{(A7)}$$

Taking the derivative of both sides of (A5) with respect to x and rearranging terms we get

$$g(x) = \mu \Pi e^{-\mu(S-x)}$$
$$+\mu \int_x^S e^{-\mu(y-x)} g(y)dy. \quad \text{(A8)}$$

Using the relationships in (A5) and (A6), we simplify (A8):

$$g(x) = \mu \Pi. \quad \text{(A9)}$$

Using (A9) and the normalizing condition in (A7), we solve for Π and $g(x)$.

$\Pi = \frac{1}{1+\mu(S-s)}$ and $g(x) = \frac{\mu}{1+\mu(S-s)}$
for $s \le x < S$.