Steady State Analysis of Continuous Review Inventory Systems: A Level Crossing Approach

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We consider a continuous review inventory system that faces two different cases of demand processes. In one case demand follows a compound Poisson process. In the other case demand is a combination of a constant deterministic component and a random component which follows a compound Poisson process. We assume negligible lead time. Orders occur when inventory level drops to or below level zero, at which point the inventory is replenished to an order-up-to level S. A level crossing approach is used to derive the steady state distribution of inventory level for each demand scenario. We use the steady state distributions to derive the total expected cost functions and determine the optimal order-up-to levels. We ran computational experiments and compared the optimal order-up-to levels from our models to the traditional EOQ. We find that the EOQ overestimates the optimal order-up-to level and in many cases is a poor approximation to the optimal.

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I. INTRODUCTION

The traditional EOQ model assumes that demand is both deterministic and uniform. Because of the simplicity of the EOQ, it is often used in practice even when demand is stochastic. We note that many operations management textbooks suggest the EOQ is a viable approach even when demand is uncertain. In this paper, we explicitly model uncertainty in demand by considering two cases. One case assumes that demand is a compound Poisson process and will be referred to as the compound Poisson case or case (1). The other case assumes demand is a combination of a constant deterministic component (as in the classical EOQ) and a random component which follows a compound Poisson process and will be referred to as the

mixture of demand case or case (2). This case represents a generalization of case (1). We note that there are practical reasons to incorporate a constant demand rate in case (2). For example a firm sells their product through two different channels. One channel is based on long term supply contracts which generate the constant component of demand. The second channel represents customers with unscheduled random orders which generates the compound Poisson process. This mixture of demand processes had been studied by other researchers. Presman and Sethi (2006) and Sobel and Zhang (2001) study this type of demand process and provide other practical examples.

We consider a continuous review inventory system and assume negligible lead time. This assumption is often made in the

literature to simplify the exposition. For example, Cetinkaya and Lee (2000), Lee and Rosenblatt (1986), and Thompstone and Silver (1975) make this assumption. Moreover, it is well known that many inventory models with non-zero lead time can be transformed to equivalent models with zero lead time as mentioned in Bassok and Anupindi (1997). The transformation from a non-zero lead time case to a zero lead time case is typically achieved by performing the analysis and setting policies that are based on inventory position (inventory on hand + on order – backlog) rather than on net inventory. In addition to simplifying the analysis, the models that consider inventory position and zero lead time can provide valuable insights that will be applicable to the non-zero lead time models. Finally, it is worthwhile to note that there are cases where lead times are indeed negligible and the standard transformation to inventory position is not necessary. For example, consider a case where the supplier is located near the demand locations and the time needed to ship and deliver the product is negligible. In such a case, it would be reasonable and realistic to consider an inventory model with zero lead time. In the conclusion, we will discuss the explicit modeling of net inventory in the presence of positive lead time as a direction for future research.

Orders occur when inventory level drops to or below level zero, at which point the inventory is replenished to an order-up-to level S. This policy has been shown to be optimal for demand processes in both cases we study here, Presman and Sethi (2006).

As is well known, the EOQ is derived from an estimate of the cost of ordering and holding inventory. In our setting, we need an estimate of the expected value of these costs. In order to do this, we need the steady state distribution of inventory levels under the order-up-to level policy as described above. We use a methodology called level crossing to derive the steady state distribution of inventory levels.

Level crossing methods were developed to obtain probability distributions in stochastic models. Brill and Posner (1977, 1981) use the level crossing approach to analyze queues. They derive the stationary probability distribution of waiting time in M/M/R queues with first come first served service discipline. Level crossing is an important component of the more general approach called the system point method. This method analyzes a stochastic process $\{I(t), t \geq 0\}$, where t represents time and I(t) is the state of the process. The state space is continuous. The process evolves over time whereby the state undergoes upward or downward jumps that occur according to a Poisson process. The upward and downward jumps are independent of each other. Moreover, the system point method examines a sample path of the process that represents a typical realization of the stochastic process over time. Based on the sample path, one usually can develop equations in terms of the stationary distribution of the system state. The derivation of the equation is intuitive and is an extension of the *rate in = rate out* principle. An overview of this method has appeared in Brill (1996, 2000).

There has been some work that extended the level crossing approach from its initial queuing application to the case of continuous review inventory models. The first such work by Azoury and Brill (1986) derives the steady state distribution of inventory level while also modeling decay in inventory level. Azoury and Brill (1992) study the case of continuous review inventory with random lead time and apply the level crossing method to derive the stationary distribution of net inventory. Brill and Chaouch (1995) consider the EOQ model with variations in demand rate at random points in time. Mohebbi and Posner (1999) use level crossing methodology to derive the stationary distribution of inventory level under continuous review with lost sales and emergency orders. They also derive exact expressions for the average cost rate functions. Berman et al (2005) consider a production/inventory model with different clearing policies and issuing policies. They derive the stationary distribution of the buffer inventory using the level crossing approach.

Other approaches to stochastic continuous review systems have considered a variety of models for the random demand process including the compound Poisson process and the general renewal process for demand arrivals with general distribution for demand sizes. For background, see Tijms (1972), Sivazlian (1974), Richards (1975), Sahin (1979), Federgruen (1983), and Altiok (1989). The analysis in these earlier works developed stationary distributions of inventory level under an assumed order policy typically of the (s,S) type. In some instances, expressions for expected cost functions and optimal policy parameters were also derived. The approaches and results have relied on the fact that orders are triggered only at demand arrival epochs. The analysis often used classical renewal theory methodology as in Sahin (1979), and Altiok (1989). As we shall see later, orders for our case (2) can be triggered at points in time other than demand arrival epochs. Therefore, using classical renewal theory is not an option for case (2).

Earlier work using level crossing was mostly focused on characterizing the steady state distribution of inventory levels and did not analyze the cost functions and the optimal policy. Preliminary work that looked at the cost functions associated with the two cases analyzed in this paper was done by Azoury and Udayabhanu (2006, 2007). This paper is an extension of that work. We have added full factorial numerical studies, sensitivity analyses, and new insights on when the classical EOQ is a good or poor approximation to the optimal. In addition, we have identified a lower bound on the total expected cost for case (2).

The rest of this paper is organized as follows. In section 2, we develop the model for the compound Poisson case and give an expression for the optimal order quantity. In section 3, we present numerical results and discuss sensitivity analysis for the compound Poisson case. In section 4, we develop the model

for the case with a mixture of demand processes, derive the steady state distribution of the inventory level, and give the expected cost function. We also present an approximation to the optimal order-up-to level and give a lower bound for the expected cost function. In section 5, we present numerical results and discuss sensitivity analysis for the case with the mixture of demand processes. In section 6, we present concluding remarks.

II. COMPOUND POISSON DEMAND

For this case, demands arrive according to an exponential distribution with rate λ and each demand size is exponential with mean $1/\mu$. Replenishment is instantaneous and replenishment orders are issued when inventory level goes down to zero. In the following we describe how the inventory level evolves over time within an order cycle. At the start of a cycle, the inventory level is S. It stays at that level for a random interval of time with mean $1/\lambda$ until the first demand occurs. At that point the inventory level drops by a random amount representing the demand size. The mean demand size is $1/\mu$. The inventory remains at that level until the next demand occurs and then drops by another random amount. This process continues until the total realized demand is greater than or equal to S, at which point, the inventory level has reached level zero or below and the cycle ends. An order is triggered to bring the inventory level up to S. In this case, the order quantity typically varies from cycle to cycle. The sample path for the inventory level is given in Fig. 1 below.

We lay out the approach we follow to solve and analyze our model. The inventory level is a stochastic process $\{I(t), t \geq 0\}$ because of the way we model the demand. We are interested in the steady state distribution of the inventory level as $t \rightarrow \infty$. To get this distribution, a typical
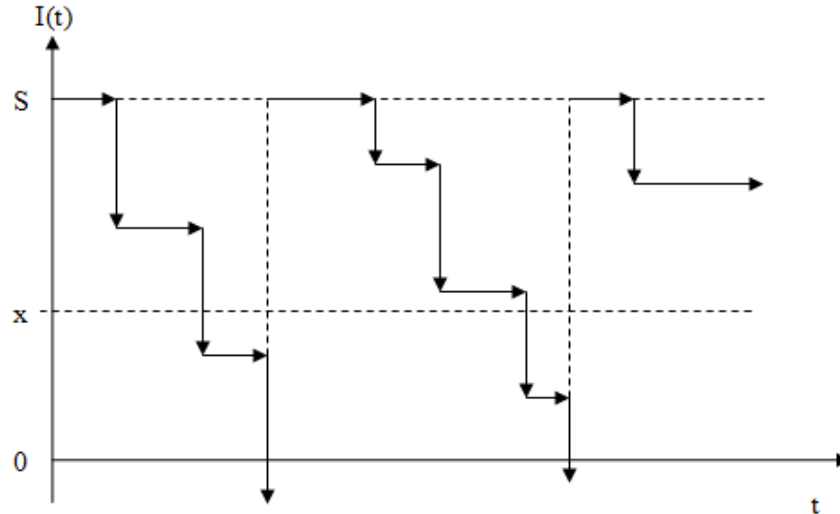
FIGURE 1. Sample Path for the Inventory Level
(Compound Poisson Demand)

sample path representing a realization of the inventory level must be generated. With the sample path in hand, the level crossing methodology can be applied to develop equations in terms of the stationary *pdf* of the inventory level. These equations must be solved to obtain the exact form of the stationary *pdf* of inventory level. One can then write the expected total cost function in terms of the stationary *pdf* and use standard optimization methods to determine the optimal order-up-to level. In the next section, we show how to use the sample path and develop the analytical results.

2.1 Derivation of the Steady State Distribution of Inventory Level

In this section, we apply the level crossing technique to derive the steady state distribution of inventory level. We use the following notation:

S = Order up to level

h = Inventory holding cost per unit of inventory per unit time

C = Fixed cost per order

λ = Demand arrival rate per unit time

$1/\mu$ = Mean demand size

g(x) = Stationary *pdf* of inventory level x

Π_S = Probability that inventory level equals S

Referring to the sample path in Fig. 1, we write model equations for Π_S and g(x) by equating up-crossing rates to down-crossing rates.

Level S is an atom with probability Π_S. The rate out is shown below on the left hand side of equation (1) as the product of the demand rate λ and the probability of being in state S. This product represents the rate of leaving state S. The inventory level returns to state S each time the inventory drops below level zero. So the rate into state S is equal to the down-crossing rate of level zero, which is shown on the right hand side of equation (1). The first term on the right hand side is the product of the demand rate and the probability of being in state S and having a demand size bigger than S, thus bringing the inventory level below zero. The second term has a similar interpretation. It is the product of the demand rate and the probability of being in any state y between zero and S and having a demand jump bigger than y.

$$\lambda\Pi_S = \lambda\Pi_S e^{-\mu S} + \lambda \int_{y=0}^{S} e^{-\mu y} g(y)dy.$$

(1)

We now equate the down-crossing and up-crossing rates for any inventory level x between 0 and S. The first term on the left hand side is the rate of down-crossing level x from state S, while the second term is the rate of down-crossing level x from any level y, where x < y < S. The right hand side, as mentioned above, is the down-crossing rate of level zero, which equals the up-crossing rate of all levels as a consequence of the replenishment policy.

$$\lambda\Pi_S e^{-\mu(S-x)} + \lambda \int_{y=x}^{S} e^{-\mu(y-x)} g(y)dy$$
$$= \lambda\Pi_S e^{-\mu S} + \lambda \int_{y=0}^{S} e^{-\mu y} g(y)dy.$$

(2)

The distribution of the inventory level must satisfy the normalizing condition shown in equation (3) below.

$$\Pi_S + \int_{y=0}^{S} g(y)dy = 1.$$

(3)

To solve for g(x), 0 < x < S, we take the derivatives of both sides of equation (2) with respect to x. This yields equation (4) below. Note that the right hand side of equation (2) is a constant, and hence, its derivative is zero.

$$\lambda\mu\Pi_S e^{-\mu(S-x)} +$$
$$\lambda\mu \int_{y=x}^{S} e^{-\mu(y-x)} g(y)dy - \lambda g(x) = 0.$$

(4)

Rearranging terms in equation (4) and using the relationships in equations (1) and (2), the density g(x) is given by

$$g(x) = \mu\Pi_S e^{-\mu(S-x)} +$$
$$\mu \int_{y=x}^{S} e^{-\mu(y-x)} g(y)dy = \mu\Pi S.$$

(5)

One can solve for Π_S and g(x), 0<x<S, by applying the relationship in equation (5) and the normalizing condition in equation (3). The continuous part of the distribution g(x) is uniform and is given by

$$g(x) = \frac{\mu}{1 + \mu S}.$$

(6)

The probability at the atom S is given by

$$\Pi_S = \frac{1}{1 + \mu S}.$$

(7)

The expected number of orders per unit time is equal to the rate into state S and is given by $\lambda\Pi_S = \frac{\lambda}{1 + \mu S}$. Therefore the cycle time between orders is equal to $\frac{1 + \mu S}{\lambda}$.

2.2 Derivation of the Expected Cost and the Optimal Order-up-to Level

The expected total cost function ETC per unit time is the sum of the expected order cost and the expected inventory holding cost and is shown below in equation (8).

$$ETC = C\lambda\Pi_S + hS\Pi_S + h\int_{0}^{S} yg(y)dy.$$

(8)

Using (6) and (7) above, we can write the ETC as follows:

$$ETC = \frac{C\lambda}{1+\mu S} + \frac{hS}{1+\mu S} + \frac{h\mu S^2}{2(1+\mu S)}.$$
(9)

The first and second derivatives of this expected cost function with respect to S are as follows:

$$\frac{\partial ETC}{\partial S} = \frac{h\mu^2 S^2}{2} + h\mu S + (h - C\lambda\mu),$$
(10)

$$\frac{\partial^2 ETC}{\partial S^2} = \frac{\mu(2\lambda\mu C - h)}{(1+\mu S)^3}.$$
(11)

The second derivative is positive when

$$2\lambda\mu C > h.$$
(12)

Thus the total expected cost function is convex in S when inequality (12) is satisfied.

The optimal S that minimizes the total expected cost can be determined by setting the first derivative of ETC in equation (10) equal to zero. The first order condition is a quadratic in S. Solving for the optimal S yields

$$S^* = \sqrt{\frac{2C\lambda}{h\mu} - \frac{1}{\mu^2}} - \frac{1}{\mu}.$$
(13)

The optimal value for S* is positive when the condition $\lambda C/h > 1/\mu$ is satisfied. If $\lambda C/h < 1/\mu$, then set S*=0. The rationale is that the inequality $\lambda C/h < 1/\mu$ implies very lumpy demand. By lumpy demand we mean infrequent arrivals (small λ values) and large demand sizes (small μ values). When demand is very lumpy, the optimal decision would be to not hold any inventory, but rather to wait until demand is realized, and then place an order.

The widely used classical EOQ model assumes a deterministic and constant demand rate D per unit time. The ratio λ/μ in the stochastic model represents the expected total demand per unit time. Thus, the classical EOQ model represents the average case scenario for our stochastic model if we set D equal to λ/μ. The EOQ model with D equal to λ/μ can be viewed as an approximation to the stochastic model that ignores the effect of randomness in demand. Note that the total cost per unit time for the classical EOQ model is:

$$TC = \frac{CD}{Q} + \frac{hQ}{2}$$
(14)

where Q represents both the order quantity and the order up to level. In this case, the optimal order up to level is given by:

$$Q^* = \sqrt{\frac{2DC}{h}}.$$
(15)

Replacing λ/μ with D in equation (13), we get

$$S^* = \sqrt{\left(\frac{2CD}{h} - \frac{1}{\mu^2}\right)} - \frac{1}{\mu},$$
(16)

which is smaller than Q*. Note that the difference is more significant for small μ. This is expected since the size of each demand jump is larger for small μ. So this suggests that the EOQ model might not be a good approximation when demand jumps are large.

III. NUMERICAL RESULTS FOR COMPOUND POISSON DEMAND

We have designed a full factorial numerical study varying three factors C/h, λ, and μ over three settings each resulting in a total of 27 trials. Examination of the total cost

expressions ETC in (9) and TC in (14) indicate that the ratio C/h is the relevant factor rather than the individual values for C (fixed cost per order) and h (holding cost per unit per unit time), hence, we fixed C at 50 and varied h. Table 1 below shows the results for the 27 trials. S^* and Q^* are the optimal order-up-to levels for the stochastic and the classical EOQ models, while ETC^* represents the optimal expected total cost under S*. $ETC(Q^*)$ gives the expected total cost of the stochastic model evaluated at the optimal EOQ, namely, Q^*. In the last column labeled % Difference, we show the percentage cost penalty from using the EOQ relative to optimal, namely, $(ETC(Q^*) - ETC^*)/ ETC^*$.

TABLE 1. Numerical Results for the Compound Poisson Demand Case

Trial	C/h	λ	μ	S*	Q*	ETC*	ETC(Q*)	% Difference
1	25	10	0.25	40.5	44.7	89.1	89.4	0.4%
2	25	20	0.25	59.1	63.2	126.2	126.5	0.2%
3	25	50	0.25	95.9	100.0	199.8	200.0	0.1%
4	25	10	0.17	48.4	54.8	108.9	109.5	0.6%
5	25	20	0.17	71.2	77.5	154.4	154.9	0.3%
6	25	50	0.17	116.3	122.5	244.6	244.9	0.1%
7	25	10	0.02	100.0	158.1	300.0	316.2	5.4%
8	25	20	0.02	167.9	223.6	435.9	447.2	2.6%
9	25	50	0.02	300.0	353.6	700.0	707.1	1.0%
10	12.5	10	0.25	27.4	31.6	125.5	126.5	0.8%
11	12.5	20	0.25	40.5	44.7	178.2	178.9	0.4%
12	12.5	50	0.25	66.6	70.7	282.4	282.8	0.2%
13	12.5	10	0.17	32.3	38.7	153.0	154.9	1.2%
14	12.5	20	0.17	48.4	54.8	217.7	219.1	0.6%
15	12.5	50	0.17	80.4	86.6	345.5	346.4	0.2%
16	12.5	10	0.02	50.0	111.8	400.0	447.2	11.8%
17	12.5	20	0.02	100.0	158.1	600.0	632.5	5.4%
18	12.5	50	0.02	194.9	250.0	979.8	1000.0	2.1%
19	5	10	0.25	15.6	20.0	196.0	200.0	2.1%
20	5	20	0.25	24.0	28.3	280.0	282.8	1.0%
21	5	50	0.25	40.5	44.7	445.4	447.2	0.4%
22	5	10	0.17	17.7	24.5	237.5	244.9	3.1%
23	5	20	0.17	28.1	34.6	341.1	346.4	1.5%
24	5	50	0.17	48.4	54.8	544.4	547.7	0.6%
25	5	10	0.02	0.0	70.7	500.0	707.1	41.4%
26	5	20	0.02	36.6	100.0	866.0	1000.0	15.5%
27	5	50	0.02	100.0	158.1	1500.0	1581.1	5.4%

The results show that as the demand pattern gets more lumpy, the performance of the EOQ deteriorates relative to the optimal. There are two parameters that contribute to lumpy demand. When μ gets smaller the demand size gets larger (more lumpy demand). When λ gets smaller, there are fewer arrivals (more lumpy demand). It is not surprising that the EOQ becomes more suboptimal when demand is more lumpy because we move away from the assumptions of the EOQ. We can see the effect of lumpiness due to μ by examining trials 19, 22, and 25. These trials have different μ values but the same C/h and λ values. Of the three trials, trial 19 has the largest μ value and has the best performance for the EOQ while trial 25 has the smallest μ value and has the worst performance. As for the effect of lumpiness due to λ, consider trials 25, 26, and 27. These trials have different λ values but the same C/h and μ values. Of the three trials, trial 27 has the largest λ value and has the best performance for the EOQ while trial 25 has the smallest λ value and has the worst performance. As noted earlier, the EOQ order-up-to level is more than the optimal S*. An examination of the derivatives of Q* and S* with respect to C/h shows that they both decrease as C/h decreases. This is expected. However, the rate of decrease for Q* is less than that of S*. Hence, when C/h gets smaller for a given λ and μ, the performance of the EOQ gets worse. In Table 2 we summarize the effect of changes to the three parameters C/h, λ, and μ on the performance of the EOQ relative to the optimal.

Table 2. Summary of Changes in the % Difference in Costs

	C/h ↓	λ ↓	μ ↓
% Difference $(ETC(Q^*) - ETC^*)/ETC^*$	↑	↑	↑

A computational comparison based on trial 16 was done to show an interesting relationship between the two cost functions (ETC andTC). The expected total cost ETC and the total cost TC were computed for values of Q ranging from 25 to 300. These cost functions are shown in Fig. 2 below. Based on these results, we note that the EOQ approach is overestimating cost for Q below Q* and underestimating cost for Q above Q* when compared to the stochastic expected cost ETC. The two costs intersect at Q*. Therefore, the only time the EOQ model measures the cost correctly is when Q = Q*. Moreover, the total EOQ cost rises less steeply than the stochastic expected total cost for high Q values and the opposite situation prevails for low Q values.

Trial 25 is worth some further comments. The parameters C/h, λ and μ were the lowest of all the trials. This is the case for the most lumpy demand with the most infrequent arrivals and the largest demand sizes. This trial also has the highest holding cost. The S* value for this case is zero. The policy calls for waiting until demand occurs and then ordering to satisfy the demand. This happens because demand is so lumpy with a high holding cost that it is not worth holding inventory in anticipation of infrequent and highly variable demand. This is clearly an extreme case but provides insight regarding the effect of the randomness of the demand process on the optimal order-up-to level.

To further illustrate the effect of lumpy demand, we ran 10 sub-trials of trial 25 where we fixed D =λ/μ = 500 by varying λ and μ as shown
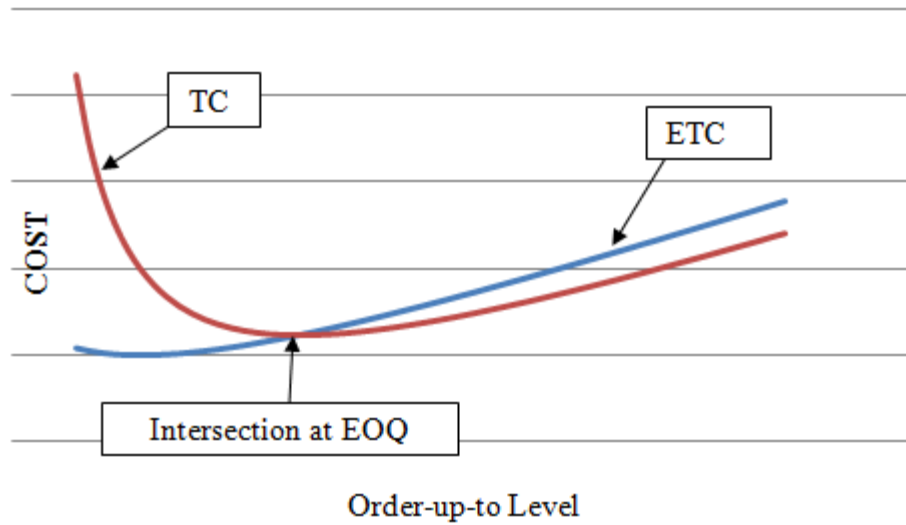
FIGURE 2. ETC and TC for Trial 16.

in Table 3 below. The first row in the table is trial 25. Moving down this table from the first row to the last row, the degree of lumpiness in demand decreases and the EOQ becomes a better estimate for the optimal order-up-to level. Moreover, the % difference in costs (ETC(Q*) − ETC*)/ETC* decreases as the lumpiness in demand decreases.

An interesting observation in Table 3 is that ETC(Q*) is the same across all sub-trials. This can be explained by what we have seen in Fig. 2, namely that ETC and TC intersect at the EOQ and that TC only depends on D which is constant across all these sub-trials.

TABLE 3. Sub-trials of Trial 25

λ	μ	S*	Q*	ETC*	ETC(Q*)	% Difference in Costs
10	0.02	0.0	70.7	500.0	707.1	41.4%
12.5	0.025	18.3	70.7	583.1	707.1	21.3%
15	0.03	29.0	70.7	623.6	707.1	13.4%
20	0.04	41.1	70.7	661.4	707.1	6.9%
25	0.05	47.8	70.7	678.2	707.1	4.3%
37.5	0.075	56.1	70.7	694.4	707.1	1.8%
50	0.1	60.0	70.7	700.0	707.1	1.0%
75	0.15	63.7	70.7	704.0	707.1	0.4%
100	0.2	65.5	70.7	705.3	707.1	0.3%
125	0.25	66.6	70.7	706.0	707.1	0.2%
250	0.5	68.7	70.7	706.8	707.1	0.0%

IV. MIXTURE OF DETERMINISTIC AND COMPOUND POISSON DEMANDS

Here we consider a continuous review inventory system where demand is a combination of two components: a constant deterministic part and a random part. The deterministic component has constant rate κ, and is identical to the modeling of demand in the classical EOQ approach. The random component follows a compound Poisson process. Demand arrivals occur exponentially with rate λ; each demand size is random and follows an exponential distribution with mean $1/\mu$. We assume negligible lead time. Orders occur when inventory level drops to or below level zero, at which point the inventory is replenished to level S. Fig. 3 below shows the sample path of inventory level. If there is no random component to the demand process, this reduces to the classical EOQ model.
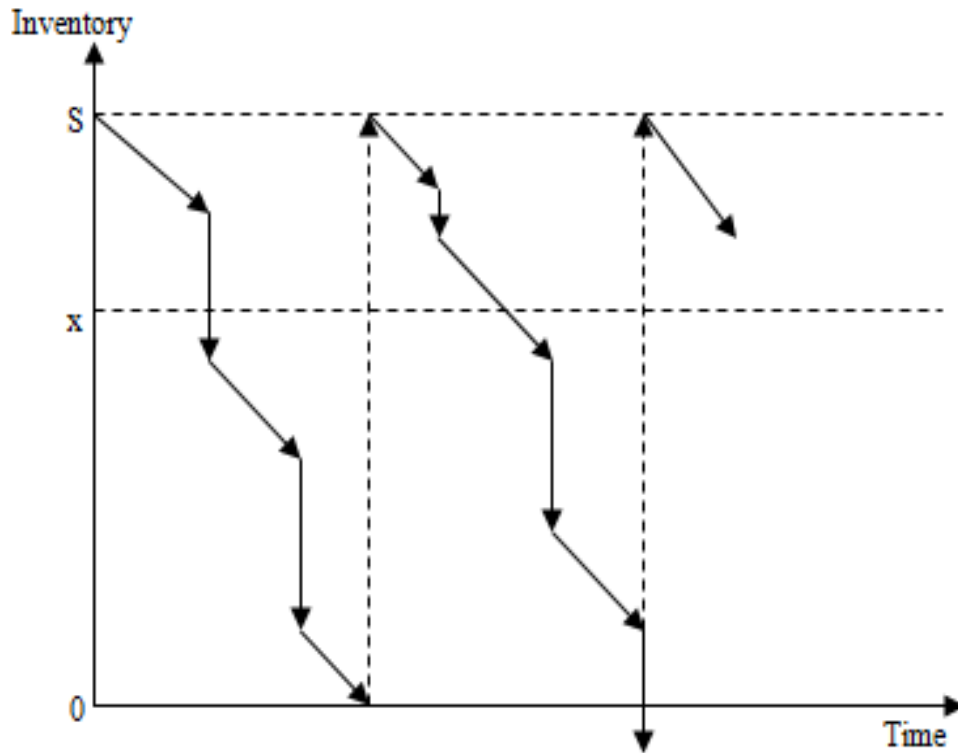

FIGURE 3. Sample Path for the Inventory Level
(Mixture of Deterministic and Compound Poisson Demands)

The deterministic component of the mixed demand processes could trigger orders at points in time other than the Poisson jump epochs. We note that such an ordering regime does not permit use of the standard renewal theory approaches to derivation of steady state distributions of inventory level. However, the level crossing method can easily handle the added complexity brought on by the deterministic component of demand and derive the needed steady state distribution of inventory level.

4.1 Derivation of the Steady State Distribution of Inventory Level

We use the same notation as in the compound Poisson case with the addition of κ = constant deterministic demand rate. Here, the density is continuous with $g(x)$ representing the

stationary pdf of inventory level x, $0<x<S$. Referring to Fig.3, the down-crossing rate of level x is

$$\kappa g(x) + \lambda \int_{x}^{S} e^{-\mu(y-x)} g(y) dy. \tag{17}$$

The up-crossing rate of level x equals the down-crossing rate of level 0, which is

$$\kappa g(0) + \lambda \int_{0}^{S} e^{-\mu y} g(y) dy. \tag{18}$$

Equating down-crossing and up-crossing rates of level x gives

$$\kappa g(x) + \lambda \int_{x}^{S} e^{-\mu(y-x)} g(y) dy$$

$$= \kappa g(0) + \lambda \int_{0}^{S} e^{-\mu y} g(y) dy. \tag{19}$$

The above equation can be solved for $g(x)$ using standard differential equations. Since the details of deriving the steady state distribution were given for the case of compound Poisson, we show the details for this case in the Appendix. This yields the stationary pdf of inventory level x as

$$g(x) = \frac{[1 + (\lambda/\kappa\mu)e^{-M(S-x)}]}{[S + (\lambda/\kappa\mu M)(1 - e^{-MS})]}$$

$$0<x<S \tag{20}$$

where $M = (\lambda/\kappa) + \mu$. Note that the density g(x) is a function of the order-up-to level S.

In Fig. 4 we show the density function g(x) for the following parameters: C=50, h=8, $\lambda = 10$, $\mu = 0.02$, and $\kappa = 100$. S is set at 40 for this illustration. While in the compound Poisson case, the density was uniform with a point mass at S, the density here is continuous and the density increases as x increases towards the upper limit S.
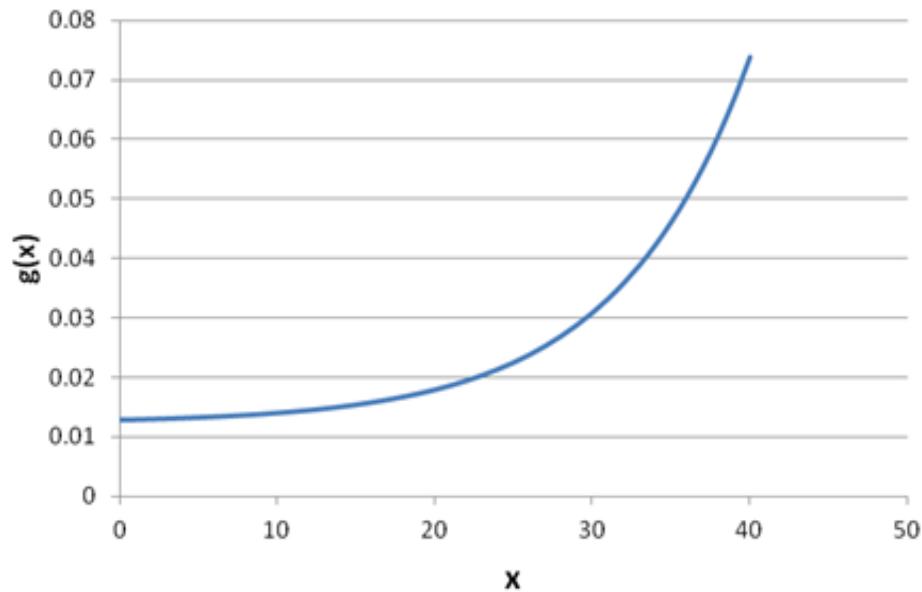

FIGURE 4. Density Function of the Inventory Level

4.2 Derivation of the Expected Cost and the Optimal Order-up-to Level

The expected total cost of ordering and holding is

$$\mathrm{E}TC = C\kappa g(S) + h\int_0^S xg(x)dx. \tag{21}$$

Substituting for g(x) in equation (21), we can write the expected total cost as

$$\mathrm{E}TC = \frac{[(C\kappa M/\mu) + h\{(S^2/2) + (\lambda/\kappa\mu M)S - (\lambda/\kappa\mu M^2)(1-e^{-MS})\}]}{[S + (\lambda/\kappa\mu M)(1-e^{-MS})]}. \tag{22}$$

Note that the cycle time in this case is given by

$$\frac{1}{\kappa g(S)} = \frac{S + (\lambda/\kappa\mu M)(1-e^{-MS})}{\kappa M/\mu}.$$

The minimization of the above cost function (22) cannot be solved in closed form. However, the optimal value S* that minimizes ETC can be determined numerically using standard optimization tools.

We develop an approximation to ETC that assumes $e^{-MS} \approx 0$. This is a reasonable assumption since both M and S are expected to be fairly large for most typical applications. By substituting 0 for e^{-MS} in the expected total cost (22), we get the following approximation

$$\mathrm{E}TC_{approx} = \frac{[(C\kappa M/\mu) + h\{(S^2/2) + (\lambda/\kappa\mu M)S - (\lambda/\kappa\mu M^2)\}]}{[S + (\lambda/\kappa\mu M)]}. \tag{23}$$

A closer inspection of ETC in (22) and ETC$_{approx}$ in (23) shows that ETC$_{approx}$ is a lower bound for ETC. Let Ŝ be the value of S that minimizes ETC$_{approx}$. We show in the Appendix that the approximate cost function ETC$_{approx}$ has a similar form to that of the classical EOQ model. By using the transformation $W = S + (\lambda/\kappa\mu M)$ in equation (23), we get the following expression for ETC$_{approx}$.

$$ETC_{approx} = \frac{(C\kappa M/\mu) - (\lambda h/\kappa\mu M^2) - (h/2)(\lambda/\kappa\mu M)^2}{W} + \frac{hW}{2}. \tag{24}$$

Note that under the condition that

$$(C\kappa M/\mu) - (\lambda h/\kappa\mu M^2) - (h/2)(\lambda/\kappa\mu M)^2 \succ 0 \tag{25}$$

the approximate cost function ETC$_{approx}$ in (24) has the same form as the total cost for the classical EOQ model if one interprets W as the order quantity, the first term as the ordering cost per unit time and the second term as the average holding cost per unit time.

The optimal W that minimizes the above approximate total cost function can be obtained by the standard square root formula typically applied for EOQ formulations. Hence, by the above transformation, the value $\hat{s}$ that minimizes the approximate cost function (24) is given by the following closed form solution.

$$\hat{S} = \sqrt{2(C\kappa M/h\mu) - (2\lambda/\kappa\mu M^2) - (\lambda/\kappa\mu M)^2} - (\lambda/\kappa\mu M). \tag{26}$$

This closed form formula assumes that the condition (25) above holds. The value for Ŝ is positive when the condition $(1 + \mu\kappa/\lambda)^2(\lambda C/h) > 1/\mu$ is satisfied. This condition is not satisfied when λ, μ, and κ are very small (very lumpy demand). This is similar to the condition in case (1) to ensure that S* is positive.

Note that the average demand rate is equal to κM/μ. We can compare Ŝ to the EOQ by substituting D for κM/μ in equation (26). We get

$$\hat{S} = \sqrt{2(CD/h) - (2\lambda/\kappa\mu M^2) - (\lambda/\kappa\mu M)^2} - (\lambda/\kappa\mu M) \tag{27}$$

which is smaller than the EOQ in (15).

V. NUMERICAL RESULTS FOR THE MIXTURE OF DEMAND PROCESSES

We have designed a full factorial numerical study varying four factors C/h, λ, μ, and κ. C/h was varied over three settings and λ, μ, and κ were varied over two settings each, resulting in a total of 24 trials. As in the previous case, the ratio C/h is the relevant factor rather than the individual values for C and h, hence, we fixed C at 50 and varied h at values 2, 8 and 10. Table 4 below shows the results for the 24 trials.

For each selection of parameter values, we performed the following:

TABLE 4. Numerical Results for the Mixture of Deterministic and Poisson Demands Case

Trial	C/h	λ	μ	κ	S*	Ŝ	Q*	ETC*	ETC(Ŝ)	ETC(Q*)	% difference in costs S-hat vs Optimal	% difference in costs EOQ vs Optimal
1	25	10	0.02	10	102.6	102.6	159.7	303.3	303.3	40.6	0.0%	5.1%
2	25	10	0.02	100	124.4	124.4	173.2	332.1	332.1	4.6	0.0%	3.3%
3	25	100	0.02	10	448.1	448.1	500.5	996.0	996.0	144.4	0.0%	0.5%
4	25	100	0.02	100	453.5	453.5	505.0	1005.0	1005.0	14.6	0.0%	0.5%
5	25	10	0.2	10	50.4	50.4	54.8	109.1	109.1	15.9	0.0%	0.3%
6	25	10	0.2	100	84.9	84.9	86.6	173.0	173.0	2.6	0.0%	0.0%
7	25	100	0.2	10	154.7	154.7	159.7	319.2	319.2	47.4	0.0%	0.0%
8	25	100	0.2	100	169.0	169.0	173.2	346.3	346.3	5.2	0.0%	0.0%
9	6.25	10	0.02	10	13.2	13.2	79.8	498.1	498.1	51.1	0.0%	27.6%
10	6.25	10	0.02	100	35.8	29.5	86.6	576.3	580.5	6.2	0.7%	16.5%
11	6.25	100	0.02	10	195.3	195.3	250.2	1961.6	1961.6	215.7	0.0%	2.1%
12	6.25	100	0.02	100	198.5	198.5	252.5	1979.9	1979.9	21.8	0.0%	2.0%
13	6.25	10	0.2	10	22.8	22.8	27.4	215.5	215.5	23.8	0.0%	1.3%
14	6.25	10	0.2	100	41.5	41.5	43.3	345.1	345.1	3.9	0.0%	0.1%
15	6.25	100	0.2	10	74.8	74.8	79.8	637.5	637.5	71.2	0.0%	0.2%
16	6.25	100	0.2	100	82.3	82.3	86.6	691.7	691.7	7.7	0.0%	0.1%
17	5	10	0.02	10	6.3	2.0	71.4	512.4	586.0	59.2	14.4%	38.6%
18	5	10	0.02	100	29.9	18.1	77.5	618.5	656.4	6.9	6.1%	20.5%
19	5	100	0.02	10	168.3	168.3	223.8	2181.7	2181.7	235.0	0.0%	2.6%
20	5	100	0.02	100	171.2	171.2	225.8	2202.3	2202.3	23.7	0.0%	2.5%
21	5	10	0.2	10	19.8	19.8	24.5	239.9	239.9	26.0	0.0%	1.6%
22	5	10	0.2	100	36.9	36.9	38.7	385.5	385.5	4.2	0.0%	0.1%
23	5	100	0.2	10	66.3	66.3	71.4	712.4	712.4	77.9	0.0%	0.2%
24	5	100	0.2	100	73.1	73.1	77.5	773.0	773.0	8.5	0.0%	0.1%

- Computed the optimal S* via a non-linear optimization of the exact total cost function ETC in (22).

- Computed the approximate order-up-to level Ŝ from the closed form formula in equation (26).

- Computed the EOQ by using $D = (\lambda / \mu) + \kappa$ as the total demand rate.

- Computed the expected total cost ETC* at the optimal S*, the expected total cost ETC(Ŝ) at the order-up-to level Ŝ, and the expected total cost ETC(Q*) at the EOQ order-up-to level Q*.

- In the second to last column in Table 4, we show the percentage of cost penalty if one uses Ŝ instead of S*, namely, (ETC(Ŝ)-ETC*)/ ETC*.

- In the last column of Table 4, we show the percentage of cost penalty if one uses Q* instead of S*, namely, (ETC(Q*) - ETC*)/ ETC*. Based on these results we make the following observations.

- In most cases Ŝ is equal to S*. In trials 10, 17, and 18, Ŝ differs from S*. For these trials the term e^{-MS} is not negligible. Since M = λ/κ + μ, M is small when λ is small, κ is large and μ is small. In trials 10 and 18 the parameters λ and μ are small and κ is large making M small and the term e^{-MS} not negligible. In trial 17, λ and μ are small and κ is also small. This makes the deterministic part of the demand very small and the stochastic part of the demand infrequent and very lumpy. For such a case the S values in the neighborhood of the optimal S value are small making the term e^{-MS} not negligible. In most cases we can conclude that ŝ is an excellent approximation to S* and is very easy to compute.

- Note that as κ increases M decreases, and for a given S, the term e^{-MS} gets larger and may not be negligible. The approximation Ŝ is derived by assuming that e^{-MS} is negligible. Therefore, at first glance, it might seem that as κ increases the performance of the approximation Ŝ will become worse. This is not the case because we have a different demand process for different values of κ and the cost function ETC depends on κ not just through M. Hence, as κ increases, the performance of the approximation Ŝ relative to the optimal S* could be better or worse. For example, trials 9 and 10 had all the same parameters expect for κ, with κ =10 for trial 9 and κ =100 for trial 10. The approximation Ŝ did worse in trial 10 (larger κ) than in trial 9 (smaller κ) . On the other hand, trials 17 and 18 also had the same parameters expect for κ, with κ =10 for trial 17 and κ =100 for trial 18. The results show an opposite pattern where the approximation in trial 18 (larger κ) performed better than in trial 17 (smaller κ).

- In all cases the EOQ is larger than S* and is not as good an approximation as Ŝ. It is especially poor for low values of μ. Note that the EOQ simplifies the modeling of the random component of demand by treating it as constant demand with rate (λ / μ) . This simplification may not have a serious impact when μ is large because demand jumps are small. However, when μ is small the simplification of an EOQ approximation can lead to large deviations from the optimal order-up-to level S*.

- The sensitivity analysis of the % difference in costs of the EOQ vs optimal with respect to changes in the parameters is similar to that of the compound Poisson case discussed earlier. In Table 5, we summarize the effect of changes to the four parameters C/h, λ, μ, and κ on the performance of the EOQ relative to the optimal.

Table 5. Summary of Changes in the % Difference in Costs

	C/h ↓	λ ↓	μ ↓	κ ↓
% Difference $(ETC(Q^*) - ETC^*)/ ETC^*$	↑	↑	↑	↑

A computational comparison based on trial 10 was done to explore the relationship between the three cost functions ETC, ETC_{approx}, and TC. In Fig. 5 below we show all three cost functions using the parameters in trial 10 for values of Q ranging from 5 to 125. First, note that the relationship between ETC and TC that we saw in the compound Poisson case does not carry over to this case. Recall that in the compound Poisson case, the two costs ETC and TC intersect at Q*. In contrast, the intersection of these two costs is not at Q*, but it is close. (Q* = 86.6. and the two costs ETC and TC intersect at 95.3). As for the relationship between ETC and ETC_{approx}, the approximate expected cost ETC_{approx} is a lower bound to ETC. Moreover, these two costs are essentially the same except at very low values of the order-up-to quantity. This relationship holds in other parameter scenarios and is not surprising given that the approximation was made by removing the term e^{-MS} which is not negligible when S is small enough.

Trial 17 is worth special attention. It has the lowest values for the four parameters and the worst performance of the Ŝ and the EOQ vs the optimal. In this trial, S* =6.3, Ŝ = 2 and Q* = 71.4. For this particular case, it is optimal to only hold inventory for the deterministic part of the demand. The random component of the demand has infrequent arrivals and large demand sizes and to a large extent, this random demand is fulfilled when it occurs rather than from inventory.
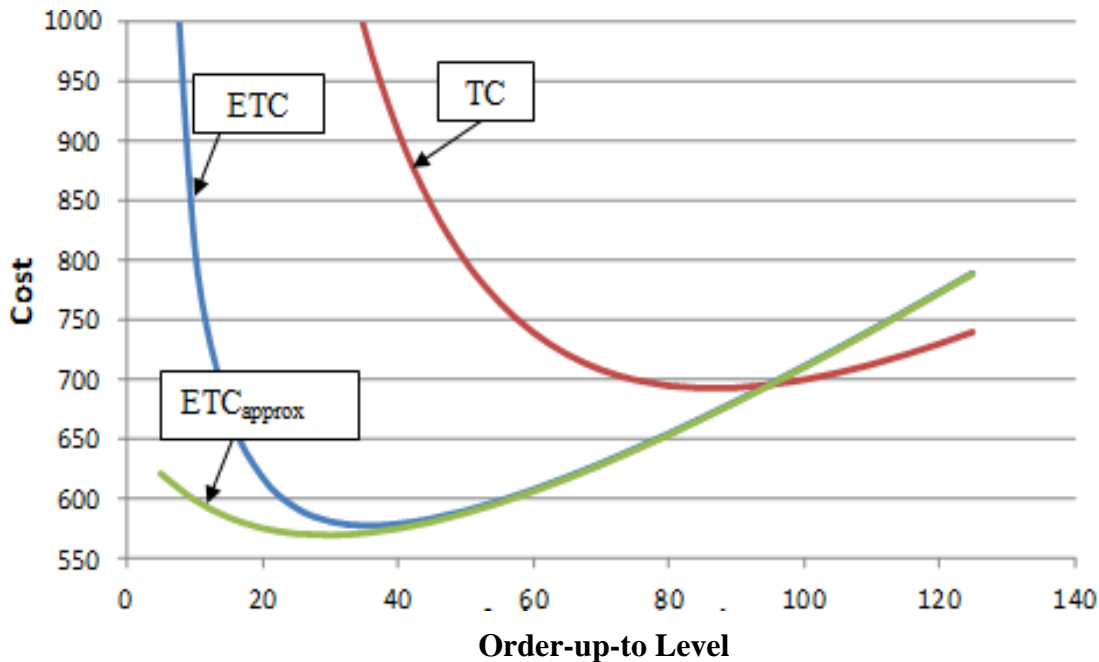

FIGURE 5. ETC, ETC_{approx} and TC for Trial 10.

TABLE 6. Sub-trials of trial 17

λ	μ	κ	S*	Ŝ	Q* (EOQ)	ETC*	ETC(Ŝ)	ETC(Q*)	% difference in costs S-hat vs Optimal	% difference in costs EOQ vs Optimal
10	0.02	10	6.3	2.0	71.4	512.4	586.0	710.2	14.4%	38.6%
12.25	0.025	20	20.8	20.8	71.4	591.8	591.8	708.7	0.0%	19.7%
14.4	0.03	30	31.8	31.8	71.4	631.9	631.9	708.2	0.0%	12.1%
18.8	0.04	40	43.9	43.9	71.4	669.2	669.2	709.4	0.0%	6.0%
23	0.05	50	50.5	50.5	71.4	685.8	685.8	710.2	0.0%	3.5%
33.75	0.075	60	58.4	58.4	71.4	701.8	701.8	711.9	0.0%	1.4%
44	0.1	70	62.1	62.1	71.4	707.2	707.2	712.7	0.0%	0.8%
64.5	0.15	80	65.5	65.5	71.4	711.1	711.1	713.4	0.0%	0.3%
84	0.2	90	67.1	67.1	71.4	712.4	712.4	713.7	0.0%	0.2%
102.5	0.25	100	68.1	68.1	71.4	713.1	713.1	713.8	0.0%	0.1%
200	0.5	110	69.8	69.8	71.4	713.9	713.9	714.1	0.0%	0.0%

To further illustrate the effect of lumpy demand, we ran 10 sub-trials of trial 17 where we fixed D =λ/μ + κ = 510 by varying λ, μ and κ as shown in the table below. The first row in the table is trial 17. Moving down this table from the first row to the last row, the degree of lumpiness in demand decreases and both Ŝ and the EOQ become better estimates for the optimal order-up-to level. Moreover, the % differences in cost for both Ŝ versus the optimal S* and the EOQ versus the optimal S* decreases as the lumpiness in demand decreases. Note that it takes just a slight decrease in demand lumpiness from Trial 17(row 1) to the sub-trial in row 2 to make Ŝ equal to S*.

VI. CONCLUSION

In this paper, we have used the level crossing approach to derive the exact form of the steady state distribution of inventory level for cases of random demand: (1) a compound Poisson process and (2) a mixture of deterministic and compound Poisson. For case (1), we derived closed form expressions for the expected total cost per unit time and the optimal order-up-to level S*. For case (2), an exact expression for the expected total cost function was also derived.

Minimization of the exact expected cost function cannot be solved in closed form, but can be minimized by a using a non-linear optimization tool to determine the optimal order-up-to level S*. We proposed an approximate expression for the expected total cost that is easy to minimize with a closed form formula that yields an approximate order-up-to level Ŝ. The cost approximation is a lower bound to the expected cost function. The approximate order-up-to level Ŝ is an excellent approximation to S* under many parameter scenarios. The approximation deteriorates in extreme scenarios of very lumpy demand

In practice, when the EOQ approach is used in an environment of random demand, the EOQ quantity is based on the average scenario for the random demand realization. To compensate for assuming the average scenario, one might tend to increase the EOQ to accommodate the randomness. However, our findings show that the EOQ overestimates the optimal order-up-to level S*. So, any attempt to increase the EOQ in order to accommodate the randomness in demand actually makes the solution worse. Therefore, we recommend using S* for case (1)

and Ŝ for case (2). Both are in closed form and very easy to compute.

In this paper, we presented results on inventory policies by using the steady state distribution of net inventory when lead time is negligible. In future work, we plan to extend this research to determine inventory policies that use the steady state distribution of net inventory with positive lead time. In this case, the policy is of the (s,S) form. We also plan to study the case of continuous replenishment (in-house production). We feel that the level crossing approach provides a promising tool to derive steady state distributions of inventory levels in these extensions.

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APPENDIX

Derivation of the steady state distribution of the inventory level for the mixture of demand case.
Equating down-crossing and up-crossing rates of level x gives

$$\kappa g(x) + \lambda \int_x^S e^{-\mu(y-x)} g(y)dy = \kappa g(0) + \lambda \int_0^S e^{-\mu y} g(y)dy. \tag{A1}$$

Taking the derivative of both sides of (A1) with respect to x and rearranging terms we get

$$g'(x) - (\frac{\lambda}{\kappa} + \mu)g(x) = -\mu g(0) - \frac{\lambda\mu}{\kappa} \int_0^S e^{-\mu y} g(y)dy. \tag{A2}$$

This is a first order differential equation and using the standard solution approach we obtain

$$g(x) = (\frac{\lambda}{\kappa} + \mu)^{-1} \left(\mu g(0) + \frac{\lambda\mu}{\kappa} \int_0^S e^{-\mu y} g(y)dy\right) + Ae^{(\frac{\lambda}{\kappa} + \mu)x} \tag{A3}$$

where A is a constant that will be determined next.

Note that equating the down-crossing and up-crossing rates for level S gives

$$\kappa g(S) = \kappa g(0) + \lambda \int_0^S e^{-\mu y} g(y)dy. \tag{A4}$$

Then

$$g(x) = (\frac{\lambda}{\kappa} + \mu)^{-1} \left(\mu g(S)\right) + Ae^{(\frac{\lambda}{\kappa} + \mu)x}. \tag{A5}$$

Replacing x with S in the above, we obtain

$$A = (\frac{\lambda}{\kappa} + \mu)^{-1} (\frac{\lambda}{\kappa}) g(S) e^{-(\frac{\lambda}{\kappa} + \mu)S}. \tag{A6}$$

Let $M = \frac{\lambda}{\kappa} + \mu$; substituting for A and M in equation (A5), we can write the density for inventory at level x as

$$g(x) = \frac{\mu g(S)}{M} \left(1 + \frac{\lambda}{\kappa\mu} e^{-M(S-x)}\right). \tag{A7}$$

The expression for g(S) is obtained using the normalizing condition $\int_0^S g(x)dx = 1$ which gives

$$g(S) = \frac{M}{\mu S + \frac{\lambda}{\kappa M}(1-e^{-MS})}.$$

So the steady state density for inventory level x, $0 \leq x \leq S$, is obtained by substituting the above expression for g(S) in (A7)

$$g(x) = \frac{[1+(\lambda/\kappa\mu)e^{-M(S-x)}]}{[S+(\lambda/\kappa\mu M)(1-e^{-MS})]}. \tag{A8}$$

Derivation of the expected total cost function.

The expected total cost of ordering and holding is

$$ETC = C\kappa g(S) + h\int_0^S xg(x)dx. \tag{A9}$$

Substituting (A8) in (A9) and applying standard integration steps including integration by parts, we can write the expected total cost as

$$ETC = \frac{[(C\kappa M/\mu) + h\{(S^2/2) + (\lambda/\kappa\mu M)S - (\lambda/\kappa\mu M^2)(1-e^{-MS})\}]}{[S+(\lambda/\kappa\mu M)(1-e^{-MS})]}. \tag{A10}$$

Derivation of the approximate closed form formula for the "order up to level"

Assuming that $e^{-MS} \approx 0$ and substituting 0 for e^{-MS} in ETC (A10), the approximate cost function is

$$ETC_{approx} = \frac{[(C\kappa M/\mu) + h\{(S^2/2) + (\lambda/\kappa\mu M)S - (\lambda/\kappa\mu M^2)\}]}{[S+(\lambda/\kappa\mu M)]}. \tag{A11}$$

Note that $\frac{1}{2}(S+\lambda/\kappa\mu M)^2 = \frac{1}{2}S^2 + \lambda S/\kappa\mu M + \frac{1}{2}(\lambda/\kappa\mu M)^2$.

Adding and subtracting $\frac{1}{2}(\lambda/\kappa\mu M)^2$ to the numerator of TC_{approx} and completing the square, we can

write the numerator as $C\kappa M/\mu - \lambda h/\kappa\mu m^2 + \frac{h}{2}(S+\lambda/\kappa\mu M)^2 - \frac{h}{2}(\lambda/\kappa\mu M)^2$.

Letting $W = S+(\lambda/\kappa\mu M)$, we get

$$ETC_{approx} = \frac{(C\kappa M / \mu) - (\lambda h / \kappa\mu M^2) - (h/2)(\lambda / \kappa\mu M)^2}{W} + \frac{hW}{2}. \qquad (A12)$$